Dynamic Bargaining with Negative Externalities

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1 The Pivotal Firm Merger Model

- N firms, $i = 0, 1, 2, ..., N-1$.
- $N-1$ possible but mutually exclusive (CS-increasing) mergers: $Mi = \{0, i\}$, $i \in 1, ..., N-1$.
- Payoffs:
  - Firm i's stand-alone flow payoff (in the absence of a merger) is u_i.
  - Following merger Mi, joint flow payoff of merged firms is u_{0i}, flow payoff of outsider j is $u_j(Mi) < u_j$.
  - Assume that each merger Mi raises bilateral profits: $u_{0i} > u_0 + u_i$.
- Time is discrete. There is an infinite horizon.
- In each period, one firm is randomly chosen to be the proposer. Firm i is chosen with probability $\lambda_i > 0$.
- Let $\delta \in (0, 1)$ denote the (common) discount factor. We are interested in the limit as $\delta \to 1$.
- We seek the stationary equilibrium.
- Let V_i denote firm i's per period value at the beginning of each period. Let $\overline{V}_i$ denote firm i's value in the limit as $\delta \to 1$.
- Let ρ_{ij} denote the unconditional probability that firm i proposes a merger to firm j, where $\sum_j \rho_{ij} \leq \lambda_i$. Let $\overline{\rho}_{ij}$ denote that probability in the limit as $\delta \to 1$.

2 Equilibrium Analysis

- Let $I = \{i | \rho_{0i} > 0 \text{ or } \rho_{i0} > 0\}$ denote the set of firms involved in a merger that occurs with positive probability.

- Let $I_0 = \{i|\rho_{0i} > 0\}$ denote the set of firms to whom firm 0 makes an offer with positive probability. Note that $I_0 \subseteq I$.

Claim 1. In stationary equilibrium, $\sum_{i>0} \rho_{0i} = \lambda_0$. Moreover, if $\rho_{0j} > 0$, then $\rho_{j0} = \lambda_j$.

Proof. We first show that $\sum_{i>0} \rho_{0i} = \lambda_0$. Suppose otherwise that $\sum_{i>0} \rho_{0i} < \lambda_0$. Consider firm $i \in I = \{i|\rho_{0i} > 0 \text{ or } \rho_{i0} > 0\}$. (It is straightforward to see that I must be non-empty as, otherwise, $V_j = u_j$ for each firm $j \in \{0, 1, ..., N-1\}$. But then each proposer j can profitably deviate by proposing to merge.) Since firm 0 must weakly prefer waiting to proposing merger Mi, we have

$$u_{0i} \leq (1-\delta)(u_0 + u_i) + \delta(V_0 + V_i).$$

However, as $i \in I$, the merger Mi does occur with positive probability, implying that we must have

$$u_{0i} \geq (1-\delta)(u_0 + u_i) + \delta(V_0 + V_i).$$

Hence,

$$u_{0i} = (1-\delta)(u_0 + u_i) + \delta(V_0 + V_i). \tag{1}$$

Firm i's value is given by

$$V_i = \rho_{i0}\left[u_{0i} - (1-\delta)u_0 - \delta V_0\right] + \sum_{j\in I, j\neq i} (\rho_{j0} + \rho_{0j})u_i(Mj)$$
$$+ \left(1 + \rho_{0i} - \sum_{j\in I}(\rho_{j0} + \rho_{0j})\right)\left[(1-\delta)u_i + \delta V_i\right]$$

But, using equation (1) and the fact that $u_i > u_i(Mj)$, this implies that $V_i \leq u_i$. Equation (1) and the assumption that $u_{0i} > u_0 + u_i$ then imply that $V_0 > u_0$. Whenever firm 0 is not the proposer (or does not propose a merger when it is the proposer), it receives $(1-\delta)u_0 + \delta V_0$, which is strictly less than V_0. Hence, for firm 0 to have a value strictly exceeding $(1-\delta)u_0 + \delta V_0$, it must propose a merger with positive probability, and the profit from proposing this merger must exceed V_0. That is, letting $I_0 = \{i|\rho_{0i} > 0\} \subseteq I$, for $j \in I_0$ we must have

$$u_{0j} - (1-\delta)u_j - \delta V_j > V_0 > (1-\delta)u_0 + \delta V_0.$$

But this contradicts equation (1), implying that $\sum_{i>0} \rho_{0i} = \lambda_0$.

To see that $\rho_{0i} > 0$ implies $\rho_{i0} = \lambda_i$, suppose otherwise that $\rho_{i0} < \lambda_i$. But this implies that equation (1) must hold for firm i, and so $V_i \leq u_i$ and $V_0 > u_0$. As $i \in I_0$, we can use the same arguments as above to show that

$$u_{0i} - (1-\delta)u_i - \delta V_i > V_0 > (1-\delta)u_0 + \delta V_0,$$

contradicting equation (1) and thus implying that $\rho_{i0} = \lambda_i$. □

- Let $\overline{I} \equiv \{i|\overline{\rho}_{0i} + \overline{\rho}_{i0} > 0\}$ and $\overline{I}_0 \equiv \{i|\overline{\rho}_{0i} > 0\}$.
- Note that if $i \in \overline{I}$, then $i \in I$ for δ arbitrarily close to 1, while the reverse does not hold. (The same holds true for the relationship between $\overline{I}_0$ and I_0.)

Claim 2. In the limit as $\delta \to 1$, the bilateral gains from any merger are non-positive in the sense that

$$u_{0i} - \left(\overline{V}_0 + \overline{V}_i\right) \leq 0, \ \forall i \in \{1, 2, ..., N-1\}.$$

If the merger occurs with positive probability in the limit, the equation holds with equality:

$$u_{0i} - \left(\overline{V}_0 + \overline{V}_i\right) = 0, \ \forall i \in \overline{I}.$$

Proof. Suppose first that $i \notin I$ for δ close to 1 (which implies that $i \notin \overline{I}$). Firm i must weakly prefer waiting over offering merger Mi. Therefore,

$$u_{0i} - (1-\delta)(u_0 + u_i) + \delta\left(V_0 + V_i\right) \leq 0,$$

which limits to

$$u_{0i} - \left(\overline{V}_0 + \overline{V}_i\right) \leq 0, \ i \notin \overline{I},$$

as asserted.

Suppose next that $i \in \overline{I}_0$, which implies that $i \in I_0$ for δ close to 1. Firm 0's value is given by

$$\begin{aligned} V_0 &= \sum_i \rho_{0i}\left[u_{0i} - (1-\delta)u_i - \delta V_i\right] + (1-\lambda_0)\left[(1-\delta)u_0 + \delta V_0\right] \\ &= \frac{\sum_i \rho_{0i}\left[u_{0i} - (1-\delta)u_i - \delta V_i\right] + (1-\lambda_0)(1-\delta)u_0}{1-\delta+\delta\lambda_0}, \end{aligned}$$

where we have used the fact (from Claim 1) that $\sum_i \rho_{0i} = \lambda_0$. Since firm 0 is indifferent between offering a merger to any firm $i \in I_0$, firm 0's value can be rewritten as

$$V_0 = \frac{\lambda_0\left[u_{0i} - (1-\delta)u_i - \delta V_i\right] + (1-\lambda_0)(1-\delta)u_0}{1-\delta+\delta\lambda_0}.$$

In the limit as $\delta \to 1$, we thus obtain

$$u_{0i} - \left(\overline{V}_0 + \overline{V}_i\right) = 0, \ \forall i \in \overline{I}_0.$$

Finally, suppose that $i \in \overline{I}$ but $i \notin \overline{I}_0$, which implies that $i \in I$ for δ close to 1. Firm i must weakly prefer offering merger Mi over waiting. Therefore,

$$u_{0i} - (1-\delta)(u_0 + u_i) + \delta\left(V_0 + V_i\right) \geq 0,$$

which limits to

$$u_{0i} - \left(\overline{V}_0 + \overline{V}_i\right) \geq 0.$$

However, as $i \notin \overline{I}_0$ (and as, from Claim 1, $\overline{I}_0$ is nonempty), $\overline{V}_0 = u_{0j} - \overline{V}_j \geq u_{0i} - \overline{V}_i$, $j \in \overline{I}_0$. Hence,

$$u_{0i} - \left(\overline{V}_0 + \overline{V}_i\right) = 0, \; \forall i \in \overline{I}.$$

□

Claim 3. Suppose firm $i \geq 1$ is such that $\sum_{j \neq i} \left(\overline{\rho}_{0j} + \overline{\rho}_{j0}\right) > 0$. Then, firm i's limiting value is given by

$$\overline{V}_i = \sum_{j \neq i} \left(\frac{\overline{\rho}_{0j} + \overline{\rho}_{j0}}{\sum_{l \neq i} \left(\overline{\rho}_{0l} + \overline{\rho}_{l0}\right)} \right) u_i(Mj).$$

Hence, by Claim 1, in the limit as $\delta \to 1$, at most one firm $i \geq 1$ can earn any rents.

Proof. We have

$$\begin{aligned} V_i &= \rho_{i0} \left[u_{0i} - (1-\delta)u_0 - \delta V_0\right] + \sum_{j \neq i} (\rho_{0j} + \rho_{j0}) u_i(Mj) \\ &\quad + \left(1 - \rho_{i0} - \sum_{j \neq i} (\rho_{0j} + \rho_{j0})\right) \left[(1-\delta)u_i + \delta V_i\right]. \end{aligned}$$

Rewriting, we obtain

$$\begin{aligned} &\left[1 - \delta + \delta \left(\rho_{i0} + \sum_{j \neq i} (\rho_{0j} + \rho_{j0})\right)\right] V_i \\ &= \rho_{i0} \left[u_{0i} - (1-\delta)u_0 - \delta V_0\right] + \sum_{j \neq i} (\rho_{0j} + \rho_{j0}) u_i(Mj) \\ &\quad + \left(1 - \rho_{i0} - \sum_{j \neq i} (\rho_{0j} + \rho_{j0})\right) (1-\delta) u_i. \end{aligned}$$

In the limit as $\delta \to 1$, the equation becomes

$$\left[\overline{\rho}_{i0} + \sum_{j \neq i} (\overline{\rho}_{0j} + \overline{\rho}_{j0})\right] \overline{V}_i = \overline{\rho}_{i0} \left[u_{0i} - \overline{V}_0\right] + \sum_{j \neq i} (\overline{\rho}_{0j} + \overline{\rho}_{j0}) u_i(Mj). \tag{2}$$

From Claim 2, we know that $\overline{\rho}_{i0} > 0$ implies $u_{0i} - \overline{V}_0 = \overline{V}_i$. But this allows us to rewrite equation (2) as

$$\left[\overline{\rho}_{i0} + \sum_{j \neq i} (\overline{\rho}_{0j} + \overline{\rho}_{j0})\right] \overline{V}_i = \overline{\rho}_{i0} \overline{V}_i + \sum_{j \neq i} (\overline{\rho}_{0j} + \overline{\rho}_{j0}) u_i(Mj),$$

i.e.,

$$\overline{V}_i = \sum_{j\neq i} \left(\frac{\overline{\rho}_{0j} + \overline{\rho}_{j0}}{\sum_{j\neq i} \left(\overline{\rho}_{0j} + \overline{\rho}_{j0}\right)} \right) u_i(Mj).$$

Note that unless $i \in I_0$, I_0 is a singleton, and $\overline{\rho}_{j0} = 0$ for all $j \neq i$, the denominator is strictly positive. (If I_0 is not a singleton, then there exists another firm $k \in I_0$, $k \neq i$. But our previous claim implies that $\overline{\rho}_{k0} = \lambda_k$.) □

- Let $h(i) \equiv \arg\max_{j\neq i} u_{0j} - u_j(Mi)$. Generically, $h(i)$ is unique. (Intuitively, firm $h(i)$ is the firm that is willing to bid most to prevent merger Mi.)

Claim 4. Suppose there exists a unique firm $k \geq 1$ such that $\overline{\rho}_{0k} + \overline{\rho}_{k0} > 0$ and $\overline{\rho}_{0i} = \overline{\rho}_{i0} = 0$ for all $i \neq k$. Then, for δ close to 1, we have $\rho_{0k} = \lambda_0$, $\rho_{k0} = \lambda_k$, $\rho_{0i} = 0$ for all $i \neq k$, and $\rho_{i0} = 0$ for all $i \neq k, h(k)$. Moreover, $u_{0k} - u_k(Mh(k)) \geq u_{0h(k)} - u_{h(k)}(Mk)$. In the limit as $\delta \to 1$, the per-period values are given by

$$\begin{aligned}
\overline{V}_k &= \min\left\{ u_k + \frac{\lambda_k}{\lambda_0 + \lambda_k}\left[u_{0k} - (u_0 + u_k)\right], u_{0k} - \left[u_{0h(k)} - u_{h(k)}(Mk)\right] \right\} \\
\overline{V}_i &= u_i(Mk),\ i \neq k \\
\overline{V}_0 &= \max\left\{ u_0 + \frac{\lambda_0}{\lambda_0 + \lambda_k}\left[u_{0k} - (u_0 + u_k)\right], u_{0h(k)} - u_{h(k)}(Mk) \right\}
\end{aligned}$$

Proof. Note first that Claim 1 implies that $\rho_{0k} = \lambda_0$, $\rho_{k0} = \lambda_k$, $\rho_{0i} = 0$ for all $i \neq k$ for δ close to 1. (To see this, note that, as $\overline{\rho}_{0k} > 0$, we must have $\rho_{k0} = \lambda_k$ for δ close to 1. We can't have $\rho_{0i} > 0$ for δ close to 1 as this would imply that $\rho_{i0} = \lambda_i$, which is inconsistent with $\overline{\rho}_{i0} = 0$. Claim 1 then implies that $\rho_{0k} = \lambda_0$ for δ close to 1.) From Claim 3 we obtain that $\overline{V}_i = u_i(Mk)$ for all $i \neq k$.

Next, we assert that $\rho_{i0} = 0$, $i \neq k, h(k)$, for δ close to 1. To see this, suppose otherwise that $\rho_{i0} > 0$ for some $i \neq k, h(k)$. As $\rho_{i0} \to 0$ as $\delta \to 1$, firm i must indifferent between offering to merge with firm 0 and waiting, i.e.,

$$u_{0i} - (1-\delta)u_0 - \delta V_0 = (1-\delta)u_i + \delta V_i,$$

which limits to

$$u_{0i} - \overline{V}_0 = u_i(Mk), \tag{3}$$

using the earlier observation that $\overline{V}_i = u_i(Mk)$. But this implies that firms 0 and firm $h(k)$ have a strict bilateral gain from merging (even in the limit as $\delta \to 1$):

$$u_{0h(k)} - \overline{V}_0 = u_{0h(k)} - [u_{0i} - u_i(Mk)] > \overline{V}_{h(k)} = u_{h(k)}(Mk),$$

where the inequality follows from the definition of $h(k)$ and the fact that $i \neq h(k)$. But this contradicts the assumption that $\overline{\rho}_{h(k)0} = 0$. Hence, $\rho_{i0} = 0$, $i \neq k, h(k)$, for δ close to 1.

We now assert that $u_{0k} - u_k(Mh(k)) \geq u_{0h(k)} - u_{h(k)}(Mk)$. To see this, suppose otherwise that

$$u_{0k} - u_k(Mh(k)) < u_{0h(k)} - u_{h(k)}(Mk).$$

However, as merger Mk happens with probability 1 in the limit, whereas merger $Mh(k)$ happens with probability 0 in the limit, we have

$$u_{0k} - \overline{V}_k \geq u_{0h(k)} - \overline{V}_{h(k)}.$$

This implies that $\overline{V}_k < u_k(Mh(k))$. In the following, we will show that this leads to a contradiction. In doing so, we distinguish between two cases.

Case (i). Suppose $\rho_{h(k)0} = 0$ for δ close to 1. Then, recalling that$\rho_{i0} = 0$ for all $i \neq k, h(k)$, we obtain

$$\begin{aligned} V_k &= \lambda_k \{u_{0k} - (1-\delta)u_0 - \delta V_0\} + (1-\lambda_k)\{(1-\delta)u_k + \delta V_k\} \\ &= \frac{1}{1-\delta+\delta\lambda_k}\{\lambda_k [u_{0k} - (1-\delta)u_0 - \delta V_0] + (1-\lambda_k)(1-\delta)u_k\} \end{aligned}$$

and

$$\begin{aligned} V_0 &= \lambda_0 \{u_{0k} - (1-\delta)u_k - \delta V_k\} + (1-\lambda_0)\{(1-\delta)u_0 + \delta V_0\} \\ &= \frac{1}{1-\delta+\delta\lambda_0}\{\lambda_0 [u_{0k} - (1-\delta)u_k - \delta V_k] + (1-\lambda_0)(1-\delta)u_0\}. \end{aligned}$$

Inserting the second equation into the first, we obtain

$$\begin{aligned} & \left\{[1-\delta+\delta\lambda_k][1-\delta+\delta\lambda_0] - \lambda_0\lambda_k\delta^2\right\} V_k \\ = \ & [1-\delta+\delta\lambda_0]\{\lambda_k [u_{0k} - (1-\delta)u_0] + (1-\lambda_k)(1-\delta)u_k\} \\ & -\delta\lambda_k\{\lambda_0 [u_{0k} - (1-\delta)u_k] + (1-\lambda_0)(1-\delta)u_0\} \\ = \ & (1-\delta)\lambda_k u_{0k} + [1-\delta+\delta\lambda_0 - (1-\delta)\lambda_k](1-\delta)u_k \\ & -\lambda_k(1-\delta)u_0. \end{aligned}$$

Hence,

$$V_k = \frac{\lambda_k u_{0k} + [1-\delta+\delta\lambda_0 - (1-\delta)\lambda_k] u_k - \lambda_k u_0}{1-\delta+\delta(\lambda_0+\lambda_k)},$$

which limits to

$$\begin{aligned} \overline{V}_k &= u_k + \frac{\lambda_k}{\lambda_0+\lambda_k}[u_{0k} - (u_0+u_k)] \\ &> u_k(Mh(k)), \end{aligned}$$

where the inequality follows from $u_k > u_k(Mh(k))$ and $u_{0k} > u_0 + u_k$. But this is a contradiction. Hence, $u_{0k} - u_k(Mh(k)) \geq u_{0h(k)} - u_{h(k)}(Mk)$ in this case.

Case (ii). Suppose $\rho_{h(k)0} > 0$ for δ close to 1. (Recall that $\rho_{i0} = 0$ for all $i \neq k, h(k)$.) As firm k can always choose not to merge, we have

$$\begin{aligned} V_k &\geq \rho_{h(k)0} u_k(Mh(k)) + \left(1-\rho_{h(k)0}\right)[(1-\delta)u_k + \delta V_k] \\ &= \frac{\rho_{h(k)0} u_k(Mh(k)) + \left(1-\rho_{h(k)0}\right)(1-\delta)u_k}{1-\delta+\delta\rho_{h(k)0}} \\ &> u_k(Mh(k)). \end{aligned}$$

Hence, in the limit as $\delta \to 1$, $\overline{V}_k \geq u_k(Mh(k))$, a contradiction. Thus, $u_{0k} - u_k(Mh(k)) \geq u_{0h(k)} - u_{h(k)}(Mk)$ in this case. To compute the values in this case (ii), note that as firm $h(k)$ must be indifferent between offering the merger and waiting (when δ is close to 1), we have

$$u_{0h(k)} - (1-\delta)\left(u_0 + u_{h(k)}\right) + \delta\left(V_0 + V_{h(k)}\right) = 0.$$

In the limit, this yields

$$\overline{V}_0 = u_{0h(k)} - \overline{V}_{h(k)} = u_{0h(k)} - u_{h(k)}(Mk).$$

Recall from above that the value of firm k is

$$V_k = \frac{1}{1-\delta+\delta\lambda_k}\left\{\lambda_k\left[u_{0k} - (1-\delta)u_0 - \delta V_0\right] + (1-\lambda_k)(1-\delta)u_k\right\},$$

which limits to

$$\overline{V}_k = u_{0k} - \overline{V}_0 = u_{0k} - \left[u_{0h(k)} - u_{h(k)}(Mk)\right].$$

Finally, to see that the value of firm 0 is the maximum of its value in cases (i) and (ii) (and thus, firm k's the respective minimum), note that firm 0 can always choose not to accept a merger offer from firm $h(k)$. □

- What can we say if there exists a firm k such that $u_{0k} - u_k(Mh(k)) > u_{0h(k)} - u_{h(k)}(Mk)$?
  - It is straightforward to construct an equilibrium where $\overline{\rho}_{0k} = \lambda_0$, $\overline{\rho}_{k0} = \lambda_k$, and $\overline{\rho}_{0i} = \overline{\rho}_{i0} = 0$ for all $i \neq k$. The equilibrium is characterized by Claim 4.
  - Is this the only equilibrium? The answer is no, not necessarily as there may exist another firm $l \neq k$ such that $u_{0l} - u_l(Mh(l)) > u_{0h(l)} - u_{h(l)}(Ml)$.
  - However, if firm k is such that $u_{0k} - u_k(Mi) > u_{0i} - u_i(Mk)$ for all $i \neq k$, then Claim 4 tells us that there cannot be an equilibrium where $\overline{\rho}_{0j} = \lambda_0$ and $\overline{\rho}_{j0} = \lambda_j$ for some $j \neq k$, and $\overline{\rho}_{0i} = \overline{\rho}_{i0} = 0$ for all $i \neq j$. Generically, we can't either have an equilibrium where there are exactly two firms, say j and l, such that $\overline{\rho}_{0i} + \overline{\rho}_{i0} > 0$ for $i \in \{j,l\}$ and $\overline{\rho}_{0i} + \overline{\rho}_{i0} = 0$ for $i \notin \{j,l\}$. (To see this, note that, in this case, $\overline{V}_j = u_j(Ml)$ and $\overline{V}_l = u_l(Mj)$. But unless $u_{0j} - u_j(Ml) = u_{0l} - u_l(Mj)$ this cannot be an equilibrium.)
  - The question then is whether we can have an equilibrium where $\overline{I}$ has at least three elements (which may or may not include firm k). Let

$$\beta_i \equiv \frac{\overline{\rho}_{0i} + \overline{\rho}_{i0}}{\sum_j \left(\overline{\rho}_{0j} + \overline{\rho}_{j0}\right)}$$

denote the probability that merger Mi occurs in stationary equilibrium. We have

$$\sum_i \beta_i = 1.$$

For any two firms $i, j \in \overline{I}$, we would need

$$\sum_{l \neq i} \frac{\beta_l}{1-\beta_i} \left[u_{0i} - u_i(Ml)\right] = \sum_{l \neq j} \frac{\beta_l}{1-\beta_j} \left[u_{0j} - u_j(Ml)\right].$$

Does such a solution with interior β's exist even if there exists a firm k such that $u_{0k} - u_k(Mi) > u_{0i} - u_i(Mk)$ for all $i \neq k$?

Example. The following numerical example shows that even if there exists a firm k such that $u_{0k} - u_k(Mi) > u_{0i} - u_i(Mk)$ for all $i \neq k$ and $u_{0k} - u_k(Mj) > u_{0i} - u_i(Mj)$ for all $i, j \neq k$, there may be an equilibrium in which $\overline{\rho}_{0i} + \overline{\rho}_{i0} > 0$ for some $i \neq k$. In the example, firm 1 takes the role of firm k and we construct a candidate equilibrium where $\overline{I} = \{1,2,3\}$. Suppose $u_{01} - u_1(M2) = 10$, $u_{01} - u_1(M3) = 5$, $u_{02} - u_2(M1) = 9$, $u_{02} - u_2(M3) = 4$, $u_{03} - u_3(M1) = 4.5$, and $u_{03} - u_3(M2) = 9.5$. Then, $\overline{V}_0 = u_{01} - \overline{V}_1 = u_{02} - \overline{V}_2 = u_{03} - \overline{V}_3$ if $\beta_1 = 0.30731$, $\beta_2 = 0.097266$, and $\beta_3 = 0.59543$. However, if one modifies the example by assuming that $u_{03} - u_3(M1) = 3$ and $u_{03} - u_3(M2) = 8$, then there does not exist a vector of probabilities $(\beta_1, \beta_2, \beta_3)$ such that $\overline{I} = \{1,2,3\}$.

Claim 5. Suppose there exists a firm k such that $u_{0k} - u_k(Mh(k)) > u_{0h(k)} - u_{h(k)}(Mk)$. Then, for δ sufficiently close to 1, there exists a stationary equilibrium such that $\overline{I} = \overline{I}_0 = \{k\}$ (and therefore characterized by Claim 4). Moreover, if $u_{0k} - u_k(Mi) > u_{0i} - u_i(Mk)$ for all $i \neq k$ and $u_i(Mk) \leq u_i(Mj)$ for all $i, j \neq k$, then this is the unique stationary equilibrium.

Proof. The part on existence is straightforward: firm 0 should optimally make an offer to firm k with maximum probability (the payoff from doing so always weakly exceeds the payoff from making an offer to any other firm), firm k should optimally make an offer to firm 0 with maximum probability (as this is better than the status quo, $V_k > u_k$, if $\rho_{h(k)0} = 0$ and better than the alternative merger $M_{h(k)}$, $V_k > u_k(Mh(k))$, if $\rho_{h(k)0} > 0$ for δ close to 1), and all other firms are weakly better off not making an offer to firm 0 (as $V_l \leq u_{0l} - V_0$) for δ close to 1, but if $\rho_{h(k)0} > 0$, then $V_{h(k)} = u_{0h(k)} - V_0$ so that firm $h(k)$ is indeed indifferent as to offering a merger.

We now prove the part on uniqueness. Suppose the assertion is wrong so that there exists an equilibrium in which either $\overline{I} = \{j\}$, $j \neq k$, or else $\overline{I}$ contains at least two elements. Suppose first that $\overline{I} = \{j\}$, where $j \neq k$. Claim4 then implies that $u_{0k} - u_k(Mi) \leq u_{0j} - u_j(Mk)$, contradicting the assumption that $u_{0k} - u_k(Mi) > u_{0i} - u_i(Mk)$ for all $i \neq k$. Hence, $\overline{I}$ must contain at least two elements.

By Claim 2, $\overline{V}_0 = u_{0i} - \overline{V}_i$ for all $i \in \overline{I}$. By Claim 3, for any $i \geq 1$,

$$u_{0i} - \overline{V}_i = \sum_{j \neq i} \frac{\beta_j}{1-\beta_i} \left[u_{0i} - u_i(Mj)\right],$$

where

$$\beta_i \equiv \frac{\overline{\rho}_{0i} + \overline{\rho}_{i0}}{\sum_j \left(\overline{\rho}_{0j} + \overline{\rho}_{j0}\right)}.$$

As $u_i(Mk) \leq u_i(Mj)$ for all $i, j \neq k$, this implies that

$$u_{0i} - \overline{V}_i \leq u_{0i} - u_i(Mk),$$

so that

$$\overline{V}_0 \leq \min_{i \in \overline{I} \setminus \{k\}} \{u_{0i} - u_i(Mk)\}. \tag{4}$$

Now,

$$\begin{aligned} u_{0k} - \overline{V}_k &= \sum_{j \neq k} \frac{\beta_j}{1-\beta_k} [u_{0k} - u_k(Mj)] \\ &> \sum_{j \neq k} \frac{\beta_j}{1-\beta_k} [u_{0j} - u_j(Mk)] \\ &\geq \min_{i \in \overline{I} \setminus \{k\}} \{u_{0i} - u_i(Mk)\} \\ &\geq \overline{V}_0, \end{aligned}$$

where the first inequality follows from the assumption that $u_{0k} - u_k(Mi) > u_{0i} - u_i(Mk)$ for all $i \neq k$, and the last inequality from equation (4). But $u_{0k} - \overline{V}_k > \overline{V}_0$ is a contradiction to Claim 2. Hence, $\overline{I} = \{k\}$. □

Remark. The sufficient condition for uniquness of equilibrium with $\overline{I} = \{k\}$ in Claim 5 is closely related to the notion that merger Mk is an "efficient and negative-externality free" state (Gomes and Jehiel, 2005). The condition that $u_i(Mk) \leq u_i(Mj)$ for all $i, j \neq k$ is indeed just the statement that merger Mk is a negative-externality free state. (In the context of mergers in the Cournot model, the condition says that merger Mk maximizes consumer surplus.) According to Gomes and Jehiel (2005), a state is efficient if it maximizes joint payoffs of all players. If merger Mk does so strictly, then in our model this amounts to saying that

$$u_{0k} + \sum_{j \neq k} u_j(Mk) > u_{0l} + \sum_{j \neq l} u_j(Ml), \ \forall l \neq k. \tag{5}$$

But this equation, in conjunction with $u_i(Mk) \leq u_i(Mj)$ for all $i, j \neq k$, implies that merger Mk bilaterally "beats" every other merger. To see this, note that the RHS of the equation (5) is weakly larger than $u_{0l} + u_l(Mk) + \sum_{j \neq l,k} u_j(Mk)$ as $u_i(Mk) \leq u_i(Mj)$ for all $i, j \neq k$. Now, this implies that

$$u_{0k} + u_l(Mk) > u_{0l} + u_l(Mk), \ \forall l \neq k.$$

But this is nothing else than our condition that merger Mk bilaterally beats every other merger. That is, the sufficient condition for uniqueness in Claim

5 is implied by the assumption that merger Mk is an efficient and negative-externality free state.[1]

- Suppose that we can find a set of firms, say $\overline{I}$, consisting of at least three elements, and a vector of probabilities (β_k) with $\sum_k \beta_k = 1$ and $\beta_k > 0$ iff $k \in \overline{I}$ such that

$$\sum_{j \neq i} \frac{\beta_j}{1-\beta_i} [u_{0i} - u_i(Mj)] = \sum_{j \neq k} \frac{\beta_j}{1-\beta_k} [u_{0k} - u_k(Mj)] \ \forall i, k \in \overline{I}.$$

The question is: Can we find a set of limiting probabilities $\overline{\rho}_{ij}$, satisfying (i) the feasibility constraints $0 \leq \sum_j \overline{\rho}_{ij} \leq \lambda_i$ and (ii) the equilibrium constraints on the $\overline{\rho}$'s implied by Claim 1, such that

$$\frac{\overline{\rho}_{0i} + \overline{\rho}_{i0}}{\sum_j \left(\overline{\rho}_{0j} + \overline{\rho}_{j0}\right)} = \beta_i.$$

Example. Suppose that there is a set of firms, say $\overline{I}$, consisting of exactly three elements, and a vector of probabilities (β_k) with $\sum_k \beta_k = 1$ and $\beta_k > 0$ iff $k \in \overline{I}$ such that

$$\sum_{j \neq i} \frac{\beta_j}{1-\beta_i} [u_{0i} - u_i(Mj)] = \sum_{j \neq k} \frac{\beta_j}{1-\beta_k} [u_{0k} - u_k(Mj)] \ \forall i, k \in \overline{I}.$$

Then, as we will now show, there exists a set of limiting probabilities $\overline{\rho}_{ij}$, satisfying (i) the feasibility constraints $0 \leq \sum_j \overline{\rho}_{ij} \leq \lambda_i$ and (ii) the equilibrium constraints on the $\overline{\rho}$'s implied by Claim 1, such that

$$\frac{\overline{\rho}_{0i} + \overline{\rho}_{i0}}{\sum_j \left(\overline{\rho}_{0j} + \overline{\rho}_{j0}\right)} = \beta_i. \tag{6}$$

Relabel firms such that

$$0 < \frac{\beta_1}{\lambda_1} \leq \frac{\beta_2}{\lambda_2} \leq \frac{\beta_3}{\lambda_3}.$$

We need to distinguish between several cases.

Case (i). Suppose $\beta_3/\beta_2 \geq (\lambda_0 + \lambda_3)/\lambda_2$. Then, set $\overline{\rho}_{03} = \lambda_0$, and thus $\overline{\rho}_{01} = \overline{\rho}_{02} = 0$ and $\overline{\rho}_{30} = \lambda_3$. Moreover, set $\overline{\rho}_{20} = (\lambda_0 + \lambda_3)\beta_2/\beta_3$ and $\overline{\rho}_{10} = \overline{\rho}_{20}\beta_1/\beta_2 = (\lambda_0 + \lambda_3)\beta_1/\beta_3$. Note that these probabilities satisfy (6) by construction. We still need to verify that $\overline{\rho}_{20} \leq \lambda_2$ and $\overline{\rho}_{10} \leq \lambda_1$. But the first inequality follows immediately from the assumption $\beta_3/\beta_2 \geq (\lambda_0 + \lambda_3)/\lambda_2$, and the second inquality from the assumption $\beta_1/\lambda_1 \leq \beta_2/\lambda_2$.

Case (ii). Suppose $\beta_3/\beta_2 < (\lambda_0 + \lambda_3)/\lambda_2$ and $\beta_3/\beta_1 \geq (\lambda_3 + \tilde{\rho}_{03})/\lambda_1$, where

$$\tilde{\rho}_{03} = \frac{(\lambda_0 + \lambda_2)\beta_3 - \lambda_3\beta_2}{\beta_2 + \beta_3}.$$

[1] I am somewhat sloppy here about whether certain inequalities have to hold strictly or weakly. But, generically, this does not matter.

(Note that $\tilde{\rho}_{03} > 0$.) Then, set $\overline{\rho}_{02} = \lambda_0 - \tilde{\rho}_{03}$, $\overline{\rho}_{03} = \tilde{\rho}_{03}$, and thus $\overline{\rho}_{02} = 0$, $\overline{\rho}_{20} = \lambda_2$ and $\overline{\rho}_{30} = \lambda_3$. Moreover, set $\overline{\rho}_{10} = (\lambda_3 + \tilde{\rho}_{03})\beta_1/\beta_3$. Note first that these probabilities satisfy (6) as

$$\begin{aligned}
\frac{\overline{\rho}_{30} + \overline{\rho}_{03}}{\overline{\rho}_{20} + \overline{\rho}_{02}} &= \frac{(\beta_2 + \beta_3)\lambda_3 + (\lambda_0 + \lambda_2)\beta_3 - \lambda_3\beta_2}{(\beta_2 + \beta_3)(\lambda_2 + \lambda_0) - (\lambda_0 + \lambda_2)\beta_3 + \lambda_3\beta_2} \\
&= \frac{(\lambda_0 + \lambda_2 + \lambda_3)\beta_3}{(\lambda_0 + \lambda_2 + \lambda_3)\beta_2} \\
&= \frac{\beta_3}{\beta_2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\overline{\rho}_{30} + \overline{\rho}_{03}}{\overline{\rho}_{10}} &= \frac{\lambda_3 + \frac{(\lambda_0+\lambda_2)\beta_3 - \lambda_3\beta_2}{\beta_2+\beta_3}}{\frac{\beta_1}{\beta_3}\left(\lambda_3 + \frac{(\lambda_0+\lambda_2)\beta_3 - \lambda_3\beta_2}{\beta_2+\beta_3}\right)} \\
&= \frac{\beta_3}{\beta_1}\left(\frac{(\beta_2 + \beta_3)\lambda_3 + (\lambda_0 + \lambda_2)\beta_3 - \lambda_3\beta_2}{(\beta_2 + \beta_3)\lambda_3 + (\lambda_0 + \lambda_2)\beta_3 - \lambda_3\beta_2}\right) \\
&= \frac{\beta_3}{\beta_1}.
\end{aligned}$$

Finally, we need to verify that $\overline{\rho}_{10} \leq \lambda_1$. But this follows immediately from the assumption that $\beta_3/\beta_1 \geq (\lambda_3 + \tilde{\rho}_{03})/\lambda_1$.

Case (iii). Suppose $\beta_3/\beta_2 < (\lambda_0 + \lambda_3)/\lambda_2$ and $\beta_3/\beta_1 < (\lambda_3 + \tilde{\rho}_{03})/\lambda_1$. Then, $\overline{\rho}_{i0} = \lambda_i$ for $i \in \{1, 2, 3\}$. Moreover,

$$\begin{aligned}
\overline{\rho}_{02} &= \beta_2(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) - \lambda_2 \\
\overline{\rho}_{03} &= \beta_3(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) - \lambda_3,
\end{aligned}$$

so that

$$\begin{aligned}
\overline{\rho}_{01} &= \lambda_0 - \overline{\rho}_{02} - \overline{\rho}_{03} \\
&= \lambda_0 + \lambda_2 + \lambda_3 - (\beta_2 + \beta_3)(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) \\
&= \beta_1(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) - \lambda_1.
\end{aligned}$$

Note that our assumption that $\beta_1/\lambda_1 \leq \beta_2/\lambda_2 \leq \beta_3/\lambda_3$ implies that if $\overline{\rho}_{01} > 0$, then $\overline{\rho}_{02} > 0$ and $\overline{\rho}_{03} > 0$. Hence, it suffices to verify that $\overline{\rho}_{01} > 0$ or, equivalently, $\overline{\rho}_{02} + \overline{\rho}_{03} < \lambda_0$, i.e.,

$$(\beta_2 + \beta_3)(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) - (\lambda_2 + \lambda_3) \quad < \quad \lambda_0,$$

or (using that $\beta_1 + \beta_2 + \beta_3$)

$$(\beta_2 + \beta_3)\lambda_1 < \beta_1(\lambda_0 + \lambda_2 + \lambda_3).$$

Rewriting, we obtain

$$\begin{aligned}\frac{\lambda_1}{\beta_1} &< \frac{\lambda_0+\lambda_2+\lambda_3}{\beta_2+\beta_3} \\ &= \frac{(\beta_2+\beta_3)\lambda_3+(\lambda_0+\lambda_2)\beta_3-\lambda_3\beta_2}{(\beta_2+\beta_3)\beta_3} \\ &= \frac{\lambda_3+\frac{(\lambda_0+\lambda_2)\beta_3-\lambda_3\beta_2}{\beta_2+\beta_3}}{\beta_3}.\end{aligned}$$

But this inequality holds by our assumption that $\beta_3/\beta_1 < (\lambda_3+\tilde{\rho}_{03})/\lambda_1$.

Define firm i's willingness to pay to prevent firm j from winning to be $W_j^i \equiv u_{oi} - u_i(j)$.

Define $j's$ maximal willingness to pay against mergers in set I as $\overline{W}^j(I) \equiv \max_{i\in I} W_i^j$ and its minimal willingess to pay against set I as $\underline{W}^j(I) \equiv \min_{i\in I} W_i^j$. Note that our results tell us that if we have a random equilibrium in which mergers in set I win with positive probability (in the limit), then $\cap_{i\in I}[\underline{W}^j(I), \overline{W}^j(I)]$ is nonempty.

Define $<_B$ as the bilateral beating relation $i <_B j \iff W_j^i < W_i^j$.

As before, define $h(i) \equiv \arg\max_j W_i^j$.

Claim 6. Suppose we have a cycle of n mergers $1, ..., n$, such that $1 = h(n)$ and $i = h(i-1)$ for $i > 1$, and $1 <_B 2 <_B ... <_B n <_B 1$. Then $\cap_{i\in I}[\underline{W}^j(I), \overline{W}^j(I)]$ is nonempty.

Proof. The nonempty intersection condition is equivalent to

$$\overline{W}^j(I) \geq \max_{i\in I, i\neq j} \underline{W}^i(I) \tag{7}$$

and

$$\underline{W}^j(I) \leq \min_{i\in I, i\neq j} \overline{W}^i(I) \tag{8}$$

for all $j \in I$. Since firm j has the highest willingess to pay against firm $j-1$ for $j = 2, ..., n$ and firm 1 has the highest willingness to pay against firm n we have:

$$\begin{aligned} W_n^1 &\geq W_n^i \text{ for } i \neq 1, n \\ W_{j-1}^j &\geq W_{j-1}^i \text{ for } i \neq j-1, j. \end{aligned}$$

Thus, for $j \neq 1$ we have

$$\overline{W}^j(I) \geq W_{j-1}^j \geq W_{j-1}^i \text{ for all } i \neq j-1, j.$$

Since $W_{j-1}^i \geq \underline{W}^i(I)$ for all $i \neq j-1, j$, this implies (7) for $j \neq 1$. A similar argument implies (7) for $j = 1$. Now consider (8) for $j \neq 1$. We have

$$\begin{aligned}
\underline{W}^j(I) &\leq \min_{i \neq j-1,j} W_i^j \\
&= \min\{W_{h(j)}^j, \min_{i \neq j-1,j,h(j)} W_i^j\} \\
&\leq \min\{\min\{W_j^{h(j)}, W_{h(j)}^{h(h(j))}\}, \min_{i \neq j-1,j,h(j)} W_i^{h(i)}\} \\
&\leq \min_{i \in I, i \neq j} \overline{W}^i(I),
\end{aligned}$$

which establishes (8) for $j \neq 1$. (The second inequality follows because $j <_B h(j)$ implies $W_{h(j)}^j < W_j^{h(j)}$, and $W_i^j \leq W_i^{h(i)}$.) Again, a similar argument holds for $j = 1$. □

Remark 7. Note that if there is no limiting equilibrium in which only one merger happens with positive probability, then there must be such a cycle.

Example. Let the payoff in the initial (pre-merger) situation give each agent $u_i = 0$. Payoffs under different mergers are:

	$M1$	$M2$	$M3$
$Firm$ 1	3	−1	−3
$Firm$ 2	−3	2	−1
$Firm$ 3	−1	−3	2

So, $W_2^1 = 4; W_3^1 = 6$, $W_1^2 = 5; W_3^2 = 3$, and $W_1^3 = 3; W_2^3 = 5$. This has a cycle, and the intersection (overlap) interval is [4,5]. Let W be the equilibrium willingness to pay of each firm. So, in a limit equilibrium there are $(\beta_1, \beta_2, \beta_3, W)$ such that $\beta_i \in [0,1]$, $\sum_i \beta_i = 1$, and

$$\begin{aligned}
\left(\frac{\beta_2}{\beta_2+\beta_3}\right)4 + \left(\frac{\beta_3}{\beta_2+\beta_3}\right)6 &= W \\
\left(\frac{\beta_1}{\beta_1+\beta_3}\right)5 + \left(\frac{\beta_3}{\beta_1+\beta_3}\right)3 &= W \\
\left(\frac{\beta_1}{\beta_1+\beta_2}\right)3 + \left(\frac{\beta_2}{\beta_1+\beta_2}\right)5 &= W.
\end{aligned}$$

Note that $\beta_i \in [0,1]$ is insured by $W \in [4,5]$. Rewriting this we want

$$\begin{aligned}
\beta_2(4-W) + \beta_3(6-W) &= 0 \\
\beta_1(5-W) + \beta_3(3-W) &= 0 \\
\beta_1(3-W) + \beta_2(5-W) &= 0.
\end{aligned}$$

Substituting for β_1 from $\sum_i \beta_i = 1$ these become:

$$\begin{aligned}
\beta_2(4-W) + \beta_3(6-W) &= 0 \\
(5-W) - \beta_2(5-W) - 2\beta_3 &= 0 \\
(3-W) + 2\beta_2 - \beta_3(3-W) &= 0.
\end{aligned}$$

Solving the first equation for $\beta_2 = -\beta_3 \left(\frac{6-W}{4-W}\right)$ and substituting we get:

$$\begin{aligned}(5-W) + \beta_3[(5-W)\left(\frac{6-W}{4-W}\right) - 2] &= 0 \\ (3-W) - \beta_3[2\left(\frac{6-W}{4-W}\right) + (3-W)] &= 0.\end{aligned}$$

The first equation says that

$$\beta_3 = -\frac{(5-W)(4-W)}{(5-W)(6-W)-2(4-W)} = \frac{(5-W)(W-4)}{(5-W)(6-W)+2(W-4)}$$

while the second says that

$$\beta_3 = \frac{(3-W)(4-W)}{2\,(6-W)+(3-W)(4-W)} = \frac{(W-3)(W-4)}{2\,(6-W)+(W-3)(W-4)}.$$

The difference is

$$\begin{aligned}\Delta &= (4-W)\left[\frac{(5-W)}{(5-W)(6-W)+2(W-4)} - \frac{(W-3)}{2\,(6-W)+(W-3)(W-4)}\right] \\ &\tilde{}(4-W)\delta(W)\end{aligned}$$

where

$$\begin{aligned}\delta(W) &= (5-W)[2\,(6-W)+(W-3)(W-4)] - (W-3)[(5-W)(6-W)+2(W-4)] \\ &= (5-W)^2(6-W) - (W-3)^2(W-4).\end{aligned}$$

Note that $\delta(W)$ is a cubic, so it has at most two values on $\mathbb{R}$ of W at which the sign of its slope changes. Now, $\delta(4) = 2$ and $\delta(5) = -4$. Moreover, $\delta'(4) = -4$ and $\delta'(5) = 4$. Together this implies that there is a single value $W^* \in [4,5]$ at which $\delta(W^*) = 0$.