Winning by a Gap*

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Abstract
We extend the standard two-player tournament contract by a gap that is imposed on the best performing agent. This agent will only receive the winner prize if he has outperformed his opponent by this gap. If neither agent wins by the gap, tournament prizes are equally shared among the agents under unverifiable performance signals and are fully kept by the principal in case of verifiable performance signals. Our results show that under several broad classes of stochastic production technologies it is not optimal for the principal to use a positive gap if agents are protected by limited liability, but for unlimited liability a positive gap can be useful to partially insure the agents. Heterogeneity of agents and correlated talents can be further arguments to introduce a positive gap. For verifiable performance signals, a positive gap is optimal under certain conditions as it reduces expected implementation costs. Moreover, in this setting the optimal tournament with a gap will dominate the optimal bonus contract if noise follows a log-concave distribution.

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1 Introduction

In tournaments, players compete for given prizes that are distributed according to relative performance.[1] Hence, the best performing player gets the highest prize, the second-best player obtains the second highest prize and so on. There are many examples of such tournaments in real life: In sports contests, individual athletes or teams try to win against other individuals or teams, respectively. In job-promotion tournaments, workers compete for a more attractive and better paid position at the next hierarchy level. Moreover, we can observe sales contests and relative performance evaluation for managers in firms as well as forced-distribution systems. In entertainment, there are diverse winner-take-all competitions like cooking and singing contests. In addition, we have litigation contests in court, R&D races in markets and rent-seeking contests. Finally, election contests in politics also have the typical characteristics of a tournament.[2]

In many situations, particularly in firms, there is a contest designer (i.e., the principal) who offers a tournament contract with optimally chosen prizes to the contestants (i.e., the agents). In our paper, we extend the usual tournament contract by a gap that is imposed on the best performing agent, who will only receive the winner prize, if he wins against his opponents by this gap. Nalebuff and Stiglitz (1983) have suggested such gap to partially insure the agents against income risk and, therefore, to improve the standard tournament contract.[3] Their result holds for homogeneous and risk averse agents in competitive markets, which yield a zero-profit condition for firms in equilibrium. Kono and Yagi (2008, p. 124) even conjecture that imposing a positive gap increases agents' incentives. On the one hand, such guess seems plausible since for given efforts of his opponents an agent must spend more

[1] The theory of tournaments and contests builds on the seminal papers by Tullock (1980), Lazear and Rosen (1981), Nalebuff and Stiglitz (1983), Malcomson (1984), O'Keeffe, Viscusi and Zeckhauser (1984).

[2] For an overview on applications of tournament models see Konrad (2009).

[3] See also McLaughlin (1988), pp. 233–234; Pietrobelli and Scarpa (1992), 87; Kalra and Shi (2001), p. 172.

effort in order to win by a gap. On the other hand, a gap may be detrimental from an incentive perspective if it leads to an overall discouragement of the contestants.

Our analysis shows that the Nalebuff-Stiglitz result on insurance analogously holds under unlimited liability of the agents, but may fail to hold, if workers are protected by limited liability. Moreover, we can show that in the standard tournament model with independent, identically distributed (i.i.d.) luck and homogeneous agents it is never optimal to introduce a positive gap, which would unambiguously discourage the contestants. A zero gap is not only optimal under i.i.d. luck but also under several broad classes of probability distributions. This finding delivers a justification for the typical modelling of tournaments without a possible gap. However, when modifying certain standard assumptions of the basic tournament model there are additional situations where a principal prefers a positive gap. Skipping the homogeneity assumption so that agents probably have different abilities, leads to settings where a positive gap can be beneficial for the principal in order to fine-tune incentives. In particular, if in a model with symmetric talent uncertainty agent heterogeneity dominates the influence of idiosyncratic shocks, the principal will be prefer a strictly positive gap. In addition, negative correlation between agents' talents favors the use of a positive gap by a principal in order to mitigate uneven competition. All these findings are based on the general assumption that agents' performance signals are unverifiable to a third party. In such situations, the principal has to pay each agent the average amount of winner and loser prizes if neither agent has won by a gap because otherwise the tournament contract would lack its important self-commitment property.[4]

As final modification of the basic tournament model we consider the case where individual performance signals are verifiable. Again, agents are homogeneous, risk neutral, protected by limited liability and have i.i.d. luck, as in the basic model. Now, under certain conditions, the optimal tournament

[4] See Malcomson (1984) on this property.

contract contains a positive gap although, for given tournament prizes, this gap reduces agents' incentives. Interestingly, optimal endogenous efforts are nevertheless higher than optimal efforts under unverifiable performance signals. This finding can be explained by the fact that the principal uses the gap to decrease expected implementation costs for a certain effort level whereas the preferred effort level is induced by the optimal tournament prizes. Due to reduced implementation costs, it is optimal for the principal to implement high effort levels. For limited liability of the agents, Oyer (2000) shows that a quota or bonus contract is optimal for the principal. We can show that the optimal tournament contract with a gap dominates the optimal bonus contract if the distribution of noise is log-concave. Thus, if a principal employs at least two agents who perform the same task (e.g., salesmen) he will optimally choose a tournament with a gap instead of bonus contracts in many situations.

Besides the work cited in the beginning of this section, our paper is related to the following recent work on contests and tournament competition. Leininger (1991) uses contest theory to model patent competition. He investigates the influence of the agents' sequential order of moves as well as the impact of different budgets on the outcome of the patent race. Krishna and Morgan (1997) analyze the war of attrition and the all-pay auction with affiliated values. They show that a war of attrition yields an expected revenue greater than or equal to that from a second-price auction and an all-pay auction, but the latter one still dominates a first-price sealed-bid auction. Dekel and Scotchmer (1999) address the problem of risk taking in winner-take-all contests in an evolutionary setting. Their results show that risk taking does not dominate individual behavior. In particular, risk taking does not endogenously evolve in contests among a small number of players. Wärneryd (2002) considers a related setting with risk taking. He finds out that both risk loving and risk averse player types can coexist in the long run. Wärneryd (2003) focuses on contests with possibly asymmetric information about the winner prize. Under certain conditions the uninformed player is more likely to win the contest than the informed player. Moldovanu and Sela (2006) compare the incentive effects of a hierarchy of elimination contests to those of a static grand contest. In particular, the latter one is optimal if the principal wants to maximize total expected efforts under linear effort costs. Yildirim (2005) and Romano and Yildirim (2005) discuss a dynamic rent-seeking contest where players can add to previous expenditures. Among other results, they can show that the Stackelberg outcome with the underdog being the leader followed by the favorite cannot be an equilibrium. Kvasov (2007) investigates the mixed-strategy equilibria of a game where two players invest a given budget in a certain number of simultaneous contests. Contrary to our paper, all these contributions do not use a contract-theoretic approach and abstract from the possibility of introducing a positive gap.

Our paper is most closely related to Ishiguro (2004) whose analysis is also based on the standard tournament setting by Lazear and Rosen (1981) and solves for the optimal tournament contract. However, the papers address different questions. Ishiguro considers the possibility of collusion between agents that compete in a tournament. Contestants gain by collectively withholding effort, which harms the principal. Hence, the principal designs a collusion-proof contract based on wage discrimination. Our paper focuses on the question under which circumstances the principal should extend the standard tournament contract by introducing a positive gap. Moreover, we assume continuous effort choices and allow for more general probability distributions. Ishiguro offers a rigorous analysis of the collusion problem by using binary efforts and i.i.d. luck.

The paper is organized as follows. The next section introduces the basic model with risk neutral and homogeneous agents that are protected by limited liability. The solution to this model is presented in Section 3. Section 4 introduces agent heterogeneity in form of symmetric talent uncertainty, whereas Section 5 skips the risk neutrality assumption of the basic model. In Section 6, we assume that the agents' performance signals are verifiable. Section 7 concludes.

2 The Basic Model

We consider a situation where a principal has to employ two agents in order to run a business. The three players are risk neutral. Agent i ($i = 1, 2$) produces an output or value for the principal that is described in monetary terms by the function[5]

$$x_i(e_i) = h(e_i) + \theta_i, \tag{1}$$

where $e_i \geq 0$ denotes agent i's effort choice with $h(0) = 0$ and $h'(e_i) > 0$, $h''(e_i) \leq 0$. Agents' outputs are also influenced by the random variables θ_1 and θ_2, which are possibly dependent and not necessarily identically distributed. We assume that $\theta_1 - \theta_2$ has a continuous density g with corresponding cdf G, which are common knowledge.[6] Note that our model comprises the standard tournament case of θ_1 and θ_2 being independent and identically distributed (i.i.d.) with density f and cdf F (see the survey by McLaughlin (1988), based on Lazear and Rosen (1981), Nalebuff and Stiglitz (1983)). In this case, we must impose the technical restriction that $\int_{-\infty}^{\infty} f^2(\theta)\, d\theta < \infty$, which ensures that g is continuous. The variables θ_1 and θ_2 denote agents' luck when competing in the tournament. For example, if the agents are salesmen their sales will crucially depend on whether the customers open the door or not, which is purely random from the three players' perspective. Alternatively, θ_i could indicate agent i's individual productivity for a certain task. The realization of $x_i(e_i)$ is observable by the principal but unverifiable by a third party. The principal can neither observe e_i (or $h(e_i)$) nor θ_i so that we have a typical moral-hazard problem.

[5] The assumption of a linear technology follows the seminal papers on tournaments by Lazear and Rosen (1981) and Nalebuff and Stiglitz (1983). McLaughlin (1988) compares both papers and concludes that the optimal gap for tournaments is the same in both settings (p. 254, endnote 7). For a use of linear technologies in labor economics see, e.g., Oyer (2000).

[6] If θ_1, θ_2 have a joint density $f(u, v)$, then the density of $\theta_1 - \theta_2$ is given by $g(u) = \int f(u + v, v)\, dv$, and if f is continuous with $f(u, v) \to 0$ for $u^2 + v^2 \to \infty$, then g is continuous. If θ_1 and θ_2 are independent with square-integrable densities f_1 and f_2, then $g(u) = \int f_1(u + v) f_2(v)\, dv$ and g is continuous.

Exerting effort e_i entails costs $c(e_i)$ for agent i with $c(0) = c'(0) = 0$ and $c'(e_i), c''(e_i) > 0$ for $e_i > 0$. In order to guarantee strict concavity of the principal's objective function, we additionally assume that

$$\frac{\partial^2}{\partial e_i^2}\left(\frac{c'(e_i)}{h'(e_i)}\right) > 0. \tag{2}$$

Note that $c'(e_i)/h'(e_i)$ is strictly increasing and that condition (2) is satisfied by a wide range of functions $c(e_i)$ and $h(e_i)$.[7] Let the agents' reservation values be normalized to $\bar{u} = 0$. Moreover, each agent is protected by limited liability in the sense that his wage payment must be non-negative. Each agent maximizes expected income, consisting of expected wage payment minus effort costs, whereas the principal maximizes expected output minus expected wage payments.

Since agents' performance signals $x_i(e_i)$ are unverifiable, explicit incentive schemes like bonuses do not work because ex post the principal would always claim that the agents have performed poorly. The agents would anticipate the principal's opportunistic behavior and optimally choose zero efforts. As pointed out by Malcomson (1984), tournament contracts that distribute a fixed amount of money according to relative performance are still feasible.[8] A tournament contract ensures that the principal will not misrepresent the agents' performance, since the full amount of money must be paid out in any case and payments are verifiable.

In this paper, we consider a modified version of a tournament contract that has been introduced by Nalebuff and Stiglitz (1983). They suggest that the better performing agent will only receive the winner prize w_H (and the other agent the loser prize w_L), if he has won by a certain gap that has been fixed in advance. If neither agent has won by a gap, the tournament prizes

[7] In particular, it holds for the family of functions $h(e_i) = \alpha e_i^{\frac{1}{m}}$ and $c(e_i) = \beta e_i^d$ with $m > 1$, $d \geq 2$ and $\alpha, \beta > 0$.

[8] More generally we could consider bonus-pool contracts as addressed by Rajan and Reichelstein (2006, 2009). However, in situations with zero reservation values and non-negative wages, the optimal bonus-pool contract is typically a winner-take-all tournament with the best performing agent obtaining the entire bonus pool.

will be equally shared (i.e., each agent receives $(w_H + w_L)/2$). The tournament contract is written as (w_H, w_L, γ) with tournament prizes satisfying the limited-liability condition $w_H, w_L \geq 0$.

The timeline is the usual one in moral-hazard models: First, the principal offers the agents a contract (w_H, w_L, γ). Then, the agents can accept or reject the contract offer. If the agents accept, they will choose a non-negative effort level. Next, the random variables θ_1 and θ_2 are realized. Finally, the principal and the agents receive their payoffs according to the contract.

3 Solution to the Basic Model

The game is solved by backwards induction: First, we consider the agents' effort choices in the tournament for a given contract (w_H, w_L, γ) under the assumption that both agents participate. Then, we derive the optimal contract (w_H^*, w_L^*, γ^*) that satisfies the agents' participation constraints, the incentive constraints and the limited-liability constraints.

At the tournament stage, agent 1 maximizes

$$\begin{aligned} EU_1(e_1) &= w_L - c(e_1) + \Delta w \cdot P(x_1(e_1) > x_2(e_2) + \gamma) \\ &\quad + \frac{\Delta w}{2} \cdot [1 - P(x_1(e_1) > x_2(e_2) + \gamma) - P(x_2(e_2) > x_1(e_1) + \gamma)] \end{aligned}$$

with $\Delta w := w_H - w_L$ denoting the spread between winner and loser prize. In any case, each agent gets at least the loser prize – either in case of losing against his opponent or as part of the winner prize – and has to pay his effort costs $c(e_i)$. If agent i wins by the gap γ, he will receive the additional prize spread Δw. If neither agent wins by the gap, each one will obtain the additional amount $\Delta w/2$. Since

$$\begin{aligned} & P(x_1(e_1) > x_2(e_2) + \gamma) = 1 - G(h(e_2) - h(e_1) + \gamma) \\ \text{and} \quad & P(x_2(e_2) > x_1(e_1) + \gamma) = G(h(e_2) - h(e_1) - \gamma) \end{aligned}$$

agent 1's objective function can be rewritten as

$$EU_1(e_1) = w_L + \frac{\Delta w}{2} [2 - G(h(e_2) - h(e_1) + \gamma) - G(h(e_2) - h(e_1) - \gamma)] - c(e_1).$$

In analogy, agent 2's objective function reads as

$$EU_2(e_2) = w_L + \frac{\Delta w}{2}\left[G\left(h(e_2) - h(e_1) - \gamma\right) + G\left(h(e_2) - h(e_1) + \gamma\right)\right] - c(e_2).$$

We assume that an equilibrium in pure strategies exists and is characterized by the first-order conditions[9]

$$\frac{\Delta w}{2}[g(h(e_2) - h(e_1) - \gamma) + g(h(e_2) - h(e_1) + \gamma)] = \frac{c'(e_1)}{h'(e_1)} = \frac{c'(e_2)}{h'(e_2)},$$

which is equivalent to

$$e_1 = e_2 =: e \quad \text{and} \quad \frac{\Delta w}{2}[g(-\gamma) + g(\gamma)] = \frac{c'(e)}{h'(e)}. \tag{3}$$

It follows that the equilibrium is symmetric, irrespective of any symmetry properties of the stochastic terms θ_1 and θ_2.

At the first stage of the game, the principal chooses the optimal contract (w_H^*, w_L^*, γ^*) that maximizes expected profit

$$2h(e) + E[\theta_1 + \theta_2] - w_L - w_H \tag{4}$$

subject to the incentive constraint (3), the limited-liability constraint $w_H, w_L \geq 0$ and the participation constraint $EU_i(e) \geq 0$. Note that the latter one is always satisfied: Each agent can ensure himself a non-negative expected utility – and, thus, at least his reservation value – by accepting any feasible contract (with non-negative tournament prizes due to the agents' limited liability) and choosing zero effort. As a direct consequence, the principal need not pay a positive loser prize to satisfy the participation constraint. Moreover, a positive w_L would increase the principal's implementation costs $w_L + w_H$ and decrease the agents' incentives (see (3)). Hence, we have $w_L^* = 0$. The resulting incentive constraint $w_H[g(-\gamma) + g(\gamma)] = 2c'(e)/h'(e)$ points

[9] The problem that the existence of pure-strategy equilibria cannot be guaranteed in general is well-known in the tournament literature; see, e.g., Lazear and Rosen (1981), p. 845, Nalebuff and Stiglitz (1983), p. 29. Existence requires a sufficiently large variance of $\theta_1 - \theta_2$ and a sufficiently convex cost function. See Wolfstetter (1999, p. 305) and Schöttner (2007) for a sufficient condition.

out that the optimal gap maximizes $g(-\gamma) + g(\gamma)$, which is the density of $|\theta_1 - \theta_2|$, because otherwise the principal could save implementation costs w_H by further increasing $g(-\gamma) + g(\gamma)$. We have thus shown the following result.

Theorem 1 *Gap γ is optimal if and only if it is a mode of the density of $|\theta_1 - \theta_2|$. In particular, the set of optimal gaps depends only on the joint distribution of θ_1 and θ_2.*

According to Theorem 1, the optimal gap does not depend on the production function h and the cost function c. Moreover, even if w_L and w_H are arbitrarily fixed the optimal gap will be the same. It solely serves to maximize incentives for a given prize spread Δw so that the implementation costs for the optimal effort level become as low as possible. The following proposition shows that under several broad classes of distributions for the θ_i the principal does not prefer to complement optimal tournament prizes with a positive gap.[10]

Theorem 2 *A zero gap is optimal under each of the following four conditions:*

a) θ_1 and θ_2 are i.i.d.

b) θ_1 and θ_2 are independent and each θ_i has a unimodal density that is symmetric about 0.

c) θ_1 and θ_2 are independent, $\theta_1 - \theta_2$ has the same distribution as $\theta_2 - \theta_1$, θ_1 or θ_2 has a log-concave density and the other variable has a unimodal density.

d) θ_1 and θ_2 have a log-concave joint density and $\theta_1 - \theta_2$ has the same distribution as $\theta_2 - \theta_1$.

If θ_1 and θ_2 are i.i.d., the optimal gap $\gamma^* = 0$ will be unique.

[10] All proofs are relegated to the appendix.

Theorem 2 points out that in many situations a positive gap will not be part of the optimal tournament contract. In particular, result a) shows that for the standard tournament case with θ_1 and θ_2 being i.i.d. the principal unambiguously rejects a positive gap, which would strictly decrease incentives. Intuitively, agents become more discouraged in the tournament the higher the gap they must beat in addition to their opponent. The technical intuition can be best seen from the results b) to d). The proof shows that in these cases density g is symmetric and has a mode at zero. The larger the gap γ the more we will go to the tails of $g(-\gamma) + g(\gamma)$ and, hence, the lower will be an agent's marginal winning probability of exerting effort in equilibrium. This negative incentive effect makes equilibrium efforts in the tournament game monotonically fall in γ.

We have already solved for the optimal loser prize, w_L^*, and the optimal gap, γ^*. Adding the optimal winner prize, w_H^*, completes the tournament contract that is chosen by the principal:

Proposition 3 *The principal chooses* $\gamma^* \in \arg\max_{\gamma}\{g(-\gamma)+g(\gamma)\}$ *and* $w_L^* = 0$. *He implements effort level* e^* *implicitly described by*

$$g(-\gamma^*) + g(\gamma^*) = \frac{c''(e^*)\,h'(e^*) - c'(e^*)\,h''(e^*)}{[h'(e^*)]^3} \tag{5}$$

via the winner prize $w_H^* = 2c'(e^*)/\left([g(-\gamma^*) + g(\gamma^*)]h'(e^*)\right)$.

Concerning the optimal tournament prizes, the principal prefers a winner-takes-all prize structure with $w_L^* = 0$ and $w_H^* > 0$. The implemented effort level may be smaller or larger than the efficient (or first-best) effort level e^{FB} that maximizes welfare $2(E[x_i(e_i)] - c(e_i))$.[11] Since the welfare function is strictly concave, e^{FB} is characterized by the first-order condition

$$1 = \frac{c'\left(e^{FB}\right)}{h'\left(e^{FB}\right)}. \tag{6}$$

The principal would always implement this solution, if there were no contractual frictions like limited-liability of the agents. To compare e^* and

[11] See also Gürtler and Kräkel (2010) on this point.

e^{FB}, we can use a parameterized example with $h(e_i) = \sqrt{e_i}$ and $c(e_i) = 0.5\kappa e_i^2$ with $\kappa > 0$. Inserting into (5) and (6) yields

$$e^* = \frac{g(-\gamma^*) + g(\gamma^*)}{6\kappa} \quad \text{and} \quad e^{FB} = \left(\frac{1}{2\kappa}\right)^{\frac{2}{3}}.$$

Depending on the magnitude of $g(-\gamma^*)+g(\gamma^*)$, we may either have $e^* < e^{FB}$ or $e^* > e^{FB}$. Similar to Lazear (1995, p. 29), we can interpret $1/[g(-\gamma^*) + g(\gamma^*)]$ as a measure of luck in the tournament. Hence, if luck has only a small impact on the outcome of the tournament the agents will have strong incentives to exert effort. In that situation, the principal has only moderate implementation costs and, thus, prefers high effort levels. Since strong incentives can lead to a fierce competition among the agents that reduces their rents, the principal may even prefer to implement more than first-best efforts: Note that an agent's rent under the optimal tournament contract and i.i.d. luck is given by[12]

$$R(e^*) = \frac{w_H}{2} - c(e^*) = \frac{c'(e^*)}{2g(0)h'(e^*)} - c(e^*).$$

We obtain

$$R'(e^*) = \frac{c''(e^*)\,h'(e^*) - c'(e^*)\,h''(e^*)}{2g(0)\,[h'(e^*)]^2} - c'(e^*) \overset{(5)}{=} h'(e^*) - c'(e^*),$$

which is negative for $e^* > e^{FB}$ due to the strict concavity of the welfare function.

4 Symmetric Talent Uncertainty and Idiosyncratic Shocks

As a first robustness check for the optimal gap derived in Theorem 2, we consider a variant of the basic model that decomposes the random variable

[12] Since workers choose identical efforts in equilibrium, each one has a winning probability of 1/2.

θ_i of the production function (1) into two components – an agent's individual talent which is only revealed after the tournament and individual luck. All other assumptions of the basic model remain unchanged. Since our focus is on the optimal gap, we will not solve for the optimal tournament prizes, which could be derived in analogy to the basic model. We assume that agent i's ($i = 1, 2$) output is described by

$$x_i(e_i) = h(e_i) + \eta_i + \sigma\epsilon_i, \tag{7}$$

where η_i and ϵ_i are random variables and σ is a positive parameter. The η_i may reflect agent i's talent or ability which is unknown to all three players (e.g., young workers often do not know their true job-specific talents when starting their careers). Since the probability distribution of η_i is common knowledge, we consider a principal-agent model with symmetric talent uncertainty in this section.[13] We do not assume that η_1 and η_2 are independent nor that they are identically distributed. Hence, we allow agents' talents to systematically differ or tend into the same direction,[14] which yields a more general setting compared to most of the previous contributions on symmetric talent uncertainty. In case of a degenerate distribution, η_1 and η_2 are deterministic, so that our model boils down to a simple unfair contest in the notion of O'Keeffe et al. (1984).

Our general setting also permits alternative interpretations of η_1 and η_2. Imagine, for example, that the two agents are salesmen that participate in a sales contest, see, e.g., Kalra and Shi (2001). In that case, η_1 and η_2 can stand for the economic environment of the salesmen, which is uncertain for all parties ex ante. There are situations where the two salesmen work in separate locations, so that η_1 and η_2 follow different distributions that depend on the characteristics of the local markets (e.g., customers' willingness to pay).

[13] See, among many others, Harris and Holmström (1982), Meyer and Vickers (1997) and Holmström (1999) on this kind of modelling, in particular Meyer (1991) and Höffler and Sliwka (2003) on tournaments with symmetric talent uncertainty.

[14] E.g., workers acquired their bachelor degrees from different universities or from the same department.

The salesmen may either sell the same product or different products, which determines how η_1 and η_2 are correlated. The correlation of the random variables may even be strictly negative. For example, if the two agents work for a financial service provider and one agent sells fixed-interest securities whereas the other one sells risky bonds then, depending on the general situation of the economy, customers will be mainly interested to buy either the former or the latter kind of financial product but not both. A common influence on the agents' sales and, thus, a positive relation between η_1 and η_2 may result from the reputation of the firm to which the salesmen belong.

Finally, the random variable ϵ_i describes individual luck or an idiosyncratic shock that is specific to agent i. The parameter σ controls the magnitude of the shocks. The larger σ the higher will be the impact of agents' individual luck on the outcome of the tournament.

Let g denote the density of the composed random variable $\eta_1 + \sigma\epsilon_1 - \eta_2 - \sigma\epsilon_2$. As shown in Section 3, for a given tournament contract, the agents choose identical efforts in a pure-strategy equilibrium described by (3). Moreover, Theorem 1 and Proposition 3 apply: The principal chooses an optimal gap that maximizes $g(-\gamma) + g(\gamma)$ and implements effort e^* described by (5). Hence, in the following we analyze the shape of the density g and its implications for the optimal gap.

If ϵ_1 and ϵ_2 are i.i.d. and σ is large, so that the influence of η_i becomes relatively small, the present model is similar to the standard tournament setting described in Section 2. It therefore seems plausible that the optimal gap should then be small. The next result confirms that this is the case under some mild conditions on the ϵ_i. It turns out that the optimal gap is even equal to zero when σ is large enough. Hence, the result can be seen as a robustness property of the optimal contract with zero gap in the standard tournament case (Theorem 2a)).

Theorem 4 *Suppose (η_1, η_2), ϵ_1 and ϵ_2 are independent and ϵ_1 and ϵ_2 have a common square-integrable density f. Suppose also that f satisfies a Lipschitz condition, which implies that f has a derivative almost everywhere, and that*

f' is integrable. Then for all σ sufficiently large, the optimal gap is zero.

Theorem 4 applies to a wide class of distributions of the noise terms ϵ_i and places no condition on the distributions of the η_i. The next result shows, on the other hand, that optimal gaps will remain strictly positive for all σ, or may even be bounded away from zero, under certain conditions on the ϵ_i and the η_i. The conditions on the η_i, equations (8) and (9), require that the difference between these variables is not too small.

Proposition 5 *a) Suppose (η_1, η_2) and (ϵ_1, ϵ_2) are independent and $\epsilon_1 - \epsilon_2$ has a density that is strictly convex on $(-\infty, 0]$ and on $[0, \infty)$. Suppose also that there exists a constant $C > 0$ such that*

$$P(\eta_1 - \eta_2 \in (-C, C)) = 0. \tag{8}$$

Then, for all $\sigma > 0$, every optimal gap $\gamma(\sigma)$ satisfies $\gamma(\sigma) \geq C$.

b) Suppose (η_1, η_2), ϵ_1 and ϵ_2 are independent and ϵ_1 and ϵ_2 have an exponential distribution with mean 1. Suppose also that

$$P(\eta_1 - \eta_2 \in (-y, y)) = o(y) \qquad (y \to 0). \tag{9}$$

Then, for all $\sigma > 0$, every optimal gap is positive.

Proposition 5 points out that the principal prefers a positive gap to fine-tune incentives if the agents are sufficiently heterogeneous. If it is very likely that agents' unknown talents η_1 and η_2 differ significantly, the tournament competition will be rather unfair. Since the probability distributions are common knowledge, the agents observe whether they face such unfair competition. In that case, the outcome of the tournament is mainly determined by the agents' talent difference. To compensate for this difference the trailing agent ("underdog") would have to spend a lot of effort. Due to his convex cost-of-effort function the underdog better reacts by giving up and saving costs. The best response of the leading agent ("favorite") then would be to choose little effort and save effort costs as well. In such situation, the principal benefits from imposing a positive gap to bring the underdog back into the race. Technically, if η_1 and η_2 sufficiently differ in the sense of (8) and (9), a large part of the probability mass will be far away from the origin. Consequently, under a zero gap agents' equilibrium efforts being implicitly described by $\Delta w g(0) = \frac{c'(e)}{h'(e)}$ (see (3)) would be rather low. The principal can work against the discouraging effect of agent heterogeneity by imposing a strictly positive gap so that $g(-\gamma) + g(\gamma) > 2g(0)$.

Not only agent heterogeneity but also idiosyncratic shocks have an influence on whether the principal should impose a positive gap or not. The following example illustrates the potential impact of dependence between the ϵ_i.

Example 6 Let η_1, η_2 be random variables satisfying condition (8) for some constant $C > 0$.

a) Suppose that ϵ_1 and ϵ_2 are i.i.d. random variables which are independent of (η_1, η_2) and that each ϵ_i has a Laplace distribution with density $\frac{1}{2}e^{-|x|}$. The Laplace density satisfies the conditions in Theorem 4, and so the optimal gap is zero when σ is large.

b) Suppose that ν_1, ν_2 and ζ are independent random variables which are independent of (η_1, η_2) and suppose that ν_1 and ν_2 have a standard normal distribution and ζ has an exponential distribution with mean 1. Let $\epsilon_i = \nu_i\sqrt{2\zeta}$, $i = 1, 2$. Then ϵ_1 and ϵ_2 are uncorrelated, but not independent, and each ϵ_i has a Laplace distribution as in a) (see also Andrews and Mellows 1974). However, a zero gap is not optimal in this case: The density of $\epsilon_1 - \epsilon_2$ is given by $2^{-3/2}e^{-|x/\sqrt{2}|}$, which is strictly convex on $(-\infty, 0]$ and on $[0, \infty)$. Thus, by Proposition 5a), every optimal gap must lie in $[C, \infty)$.

By using a standard normal distribution for the noise variables, we can highlight the interplay of agent heterogeneity and idiosyncratic noise for the optimal gap:

Proposition 7 *Suppose $(\eta_1, \eta_2), \epsilon_1, \epsilon_2$ are independent and ϵ_1 and ϵ_2 are standard normal variables. If $|\eta_1 - \eta_2| \leq \sqrt{2}\sigma$ almost surely, then the opti-*

mal gap is zero. If $|\eta_1 - \eta_2| > \sqrt{2}\sigma$ *almost surely, then every optimal gap is positive.*

According to Proposition 7, the zero-gap result of Section 3 is restored under a moderate degree of agent heterogeneity and a sufficiently high impact of idiosyncratic noise. However, if tournament competition is mainly determined by agent heterogeneity, we will have the opposite result leading to a strictly positive optimal gap.

The following example addresses the extreme case of absolutely contrarian talents of the agents. Such constellation sketches a personnel policy of a firm that is based on perfect hedging by recruiting its workforce from opposite labor pools (e.g., mathematicians and social scientists). If η_1 does not reflect an agent's talent but his economic environment the following example will allow an alternative interpretation. In particular, as mentioned in the beginning of this section, the two agents may work for a financial service provider with one agent selling fixed-interest securities and the other agent risky bonds. Under either interpretation of η_1, given the following kind of heterogeneity, there are many situations in which the optimal gap is zero.

Example 8 Suppose that

$$x_1(e_1) = h(e_1) + \eta_1 + \sigma\epsilon_1, \qquad x_2(e_2) = h(e_2) - \eta_1 + \sigma\epsilon_2, \tag{10}$$

and that ϵ_1, ϵ_2 are standard normal variables, and $\eta_1, \epsilon_1, \epsilon_2$ are independent. Then, by Proposition 7, the optimal gap is zero if $|\eta_1| \leq \sigma/\sqrt{2}$ almost surely.

Under (10), a zero gap is also optimal if $\eta_1, \epsilon_1, \epsilon_2$ are independent, ϵ_1 and ϵ_2 have the same unimodal density and η_1 has a symmetric log-concave density. These assumptions imply that $\sigma(\epsilon_1 - \epsilon_2)$ is unimodal and that $2\eta_1$ is strongly unimodal, so that the density of $2\eta_1 + \sigma(\epsilon_1 - \epsilon_2)$ is unimodal, see Dharmadhikari and Joag-dev (1988), pp. 15–20. As this density is symmetric, it assumes its maximum at 0, so that a zero gap is optimal.

The example illustrates that we are back to the zero-gap result of the basic model, if idiosyncratic luck dominates the influence of agent heterogeneity. This finding even holds in the case where agents have uncertain but completely contrarian talents or sell different products that are perfect substitutes from the customers' point of view.

We can use a binary talent distribution in order to stress the interplay of agents' talents, idiosyncratic shocks and talent correlation:

Proposition 9 *Suppose in (7),* $(\eta_1, \eta_2), \epsilon_1, \epsilon_2$ *are independent,* ϵ_1 *and* ϵ_2 *are standard normal variables, and*

$$P(\eta_i = a) = p, \qquad P(\eta_i = 0) = 1 - p, \qquad i = 1, 2,$$

where $a > 0$ *and* $0 < p < 1$. *Let* ϱ *denote the correlation coefficient of* η_1 *and* η_2. *If* $(p - p^2)(1 - \varrho) \leq \frac{1}{3}$ *or* $a \leq \sqrt{2}\sigma$, *then the optimal gap is zero. If* $(p - p^2)(1 - \varrho) > \frac{1}{3}$ *and* a *is so large that*

$$\frac{1}{3 - 2e^{-a^2/(4\sigma^2)}} \leq (p - p^2)(1 - \varrho), \tag{11}$$

then the optimal gap is positive.

Remark 10 a) If $\varrho \geq 0$, then $(p - p^2)(1 - \varrho) \leq \frac{1}{4}$. Thus, the optimal gap can only be positive if η_1 and η_2 are negatively correlated. b) Suppose $(p - p^2)(1 - \varrho) > \frac{1}{3}$ and let $a_0 > 0$ be such that there is equality in (11) for $a = a_0$. Then the optimal gap is zero for $a \leq \sqrt{2}\sigma$ and positive for $a \geq a_0$. We conjecture that there is a threshold $a_1 \in [\sqrt{2}\sigma, a_0]$ so that the optimal gap is zero for $a < a_1$ and positive for $a > a_1$.

Proposition 9 offers a nice summary of the main effects that determine the optimal gap. Large values of a and $p - p^2$ and, hence, a high risk $Var(\eta_i) = a^2(p - p^2)$ of the agents' talent distribution make inequality (11) more easy to be satisfied, so that the principal prefers a positive gap. In this situation, a heterogeneous match between a rather strong favorite (i.e., an agent with large talent a) and a rather weak underdog (i.e., an agent with zero talent) becomes very likely, which makes a countervailing positive gap quite attractive for the principal. However, if idiosyncratic noise has a deep impact (i.e., σ is large) and dominates the influence of the talent distribution, a zero gap will be optimal. Finally, if agents' unknown talents are positively correlated or agents' performance is influenced by common factors like firm reputation, tournament competition will filter out part of the stochastic impact on agents' relative performance, which again favors a zero gap from the principal's viewpoint.

5 Risk Aversion

In this section, we use the production technology of the basic model (see (1)) and consider the standard tournament case with θ_1 and θ_2 being i.i.d., but skip the assumption of risk neutral agents. Instead, we assume agents to be risk averse. Nalebuff and Stiglitz (1983) and McLaughlin (1988) considered tournaments with a gap under risk aversion. However, we depart from the two papers as we do not assume competitive markets that lead to zero expected profits for firms in equilibrium. We keep the assumption of the basic model that the principal has all the bargaining power and the agents face a certain reservation value, which describes their best contract alternative in the market. In the following, we differentiate between two cases. First, agents are assumed to be unlimitedly liable so that the principal can charge positive entrance fees (i.e., fix negative loser prizes $w_L < 0$). As second case, again agents are protected by limited liability and must earn non-negative wages as in the basic model.[15]

An agent's utility of earning wage w and exerting effort e is now described by

$$U(w,e) = u(w) - c(e) \tag{12}$$

with $u(w)$ being monotonically increasing and strictly concave, satisfying $u(0) = 0$, and $c(e)$ as disutility of effort having the same technical characteristics as function $c(e_i)$ in the basic model. Hence, we have $U(0,0) = 0$.

[15] Of course, introducing unlimited liability in the basic model would eliminate any relevant contractual friction so that the principal unambiguously implements first-best effort.

Let again each agent's reservation utility be normalized to $\bar{U} = 0$.

Given the tournament contract (w_H, w_L, γ), at stage 2 agent 1 now maximizes

$$EU_1(e_1) = u(w_H) \cdot [1 - G(h(e_2) - h(e_1) + \gamma)] + u(w_L) \cdot G(h(e_2) - h(e_1) - \gamma)$$
$$+u\left(\frac{w_H + w_L}{2}\right) \cdot [G(h(e_2) - h(e_1) + \gamma) - G(h(e_2) - h(e_1) - \gamma)] - c(e_1)$$

with G denoting the cdf of $\theta_1 - \theta_2$ with density g. The corresponding first-order condition is given by

$$u(w_H) \cdot g(h(e_2) - h(e_1) + \gamma) - u(w_L) \cdot g(h(e_2) - h(e_1) - \gamma)$$
$$+u\left(\frac{w_H + w_L}{2}\right) \cdot [g(h(e_2) - h(e_1) - \gamma) - g(h(e_2) - h(e_1) + \gamma)] = \frac{c'(e_1)}{h'(e_1)}.$$

Following Nalebuff and Stiglitz (1983) and McLaughlin (1988) by restricting the analysis to symmetric, pure-strategy Nash equilibria with $e_1 = e_2 = e$ yields identical efforts of both homogeneous players that are implicitly described by[16]

$$[u(w_H) - u(w_L)] \cdot g(\gamma) = \frac{c'(e)}{h'(e)} \tag{13}$$

with $g(\gamma) = g(-\gamma)$ due to symmetry of the convolution. By inserting the symmetry condition $e_1 = e_2 = e$ in the agents' objective functions, each agent's participation constraint reads as

$$[u(w_H) + u(w_L)] \cdot G(-\gamma) + u\left(\frac{w_H + w_L}{2}\right) \cdot [1 - 2G(-\gamma)] - c(e) \geq 0. \tag{14}$$

If agents are unlimitedly liable, the principal maximizes (4) subject to the incentive constraint (13) and the participation constraint (14); if agents are protected by limited liability, the principal faces the additional constraint $w_H, w_L \geq 0$. For either case, we will not solve for the optimal tournament contract (w_H^*, w_L^*, γ^*), but focus on the question whether the optimal gap γ^* is positive or zero.

[16] Since agents are completely homogeneous, symmetric equilibria seem to be most plausible. Note that the symmetric equilibrium is unique. However, in our general setting we cannot exclude the existence of additional asymmetric equilibria.

Under *unlimited liability*, the principal chooses w_L^* to make (14) just bind; any higher value of the loser prize would only decrease incentives and increase expected labor costs. In the following, we can state a sufficient condition for the optimal gap to be positive. First, note that switching from $\gamma = 0$ to a positive gap $\gamma > 0$ unambiguously relaxes the participation constraint since $u(\cdot)$ is strictly concave:

$$\frac{\partial}{\partial \gamma}\left(\left[u(w_H) + u(w_L)\right] \cdot G(-\gamma) + u\left(\frac{w_H + w_L}{2}\right) \cdot \left[1 - 2G(-\gamma)\right]\right)\Bigg|_{\gamma=0}$$
$$= 2g(0)\left[u\left(\frac{w_H + w_L}{2}\right) - \frac{u(w_H) + u(w_L)}{2}\right] > 0.$$

When switching from $\gamma = 0$ to $\gamma > 0$ in the incentive constraint, we must keep in mind that the convolution $g(\cdot)$ is not necessarily differentiable at zero.[17] Thus, we take the right derivative $d_+ \, g(\gamma)|_{\gamma=0}$ in (13) when implicitly differentiating optimal effort e with respect to γ at $\gamma = 0$:

$$\left.\frac{\partial e}{\partial \gamma}\right|_{\gamma=0} = \frac{\left[u(w_H) - u(w_L)\right] \cdot d_+ \, g(\gamma)|_{\gamma=0}}{c'(e)/h'(e)}. \tag{15}$$

Since the convolution g has its global maximum at zero,[18] we either have $d_+ \, g(\gamma)|_{\gamma=0} = g'(0) = 0$ if g is differentiable at zero, or $d_+ \, g(\gamma)|_{\gamma=0} < 0$ otherwise. Hence, switching from a zero gap to a marginally positive gap is always optimal if g is differentiable at zero (e.g., if θ_1, θ_2 and, therefore,[19] $\theta_1 - \theta_2$ are normally distributed). In that case, both w_H and w_L can be adjusted downward to leave the incentive constraint (13) unchanged (i.e., $u(w_H) - u(w_L)$ remains constant) and make the participation constraint (14) again bind. The principal would prefer such switch in order to implement a certain (optimal) effort level at lower costs $w_H + w_L$. If $d_+ \, g(\gamma)|_{\gamma=0} < 0$, under a marginally positive gap the principal can implement the same effort level as under a zero gap by appropriately reducing w_L. However, this reduction is

[17] If, for example, θ_1 and θ_2 are independent and exponentially distributed the convolution g will be a Laplace density (see, e.g., Feller 1966, pp. 48-49, Ross 2003, p. 333), which is not differentiable at its mode.

[18] See the proof of Proposition 2a).

[19] See, for example, Wolfstetter (1999), p. 306.

only feasible if a marginally positive gap sufficiently relaxes the participation constraint (14). Consequently, if $-d_+ \, g(\gamma)|_{\gamma=0}$ is not too large the optimal tournament contract will include a strictly positive gap.[20]

Intuitively, the higher the gap the more the agents are insured against income risk: In the symmetric equilibrium under a zero gap, each agent gets the high utility $u(w_H)$ with probability 1/2 and the low utility $u(w_L)$ with the same probability. If the principal imposes a positive gap, part of the probability mass is shifted to the event that neither agent wins by a gap and receives the intermediate utility $u\left(\frac{w_H+w_L}{2}\right)$. In the extreme case that the principal chooses $\gamma = \bar{\gamma}$ with $\bar{\gamma}$ being so large that $G(-\bar{\gamma}) = 0$, both agents are even perfectly insured against income risk since each one gets $u\left(\frac{w_H+w_L}{2}\right)$ for sure. However, from the incentive constraint we can see that – for given tournament prizes – implemented efforts decrease when switching from a zero gap to a positive one. Altogether, if insurance advantages dominate incentive disadvantages when deviating from a zero gap, the principal will prefer a strictly positive gap.

Under *limited liability*, we have qualitatively the same result as in the basic model: since the agents have zero reservation utilities, they will accept any feasible tournament contract with non-negative prizes as they can guarantee themselves non-negative expected utilities by choosing zero effort. Hence, the principal does not have to care for the participation constraint when solving for the optimal contract. Moreover, we have $w_L^* = 0$ since positive loser prizes decrease incentives and increase implementation costs. The optimal gap therefore maximizes the left-hand side of (13), yielding $\gamma^* = 0$, analogously to Theorem 2a). The following proposition summarizes our results:

Theorem 11 *Let the agents be risk averse with utility function (12). If agents are unlimitedly liable, there exists a cut-off* $\hat{g} > 0$ *so that* $\gamma^* > 0$ *if*

[20] Note that this condition is sufficient but not necessary: (14) is more relaxed the higher γ, but the convolution may be non-monotonic in the positive domain. Hence, instead of a marginally positive gap the principal may prefer an even higher value of γ.

$-d_+ \, g(\gamma)|_{\gamma=0} < \hat{g}$. *If agents are protected by limited liability, the optimal gap will be zero* ($\gamma^* = 0$)*.*

6 Verifiable Performance Signals

6.1 Optimal Tournament Contract

If agents' outputs $x_i(e_i)$ are verifiable performance signals, the principal is not forced to pay out the tournament prizes in any case in order to guarantee the self-commitment property of the tournament scheme. Now he can withhold the winner prize w_H if neither agent has won by gap γ. We keep the main assumptions of the basic model with i.i.d. luck, in particular, risk neutral agents, limited liability and zero reservation values. Therefore, again the principal offers a zero loser prize since the participation constraint is satisfied under any feasible tournament contract with $w_H \geq 0$ and because a positive loser prize would only decrease the agents' incentives and increase the principal's implementation costs. To avoid technical problems, we impose the restrictions that the principal can only choose a finite gap[21] $\gamma \in [0, \bar{\gamma}]$ with $\bar{\gamma} > 0$ and that the densities are strictly positive on this interval.

At the tournament stage, agent i ($i = 1, 2$) maximizes

$$w_H \cdot P(x_i(e_i) > x_j(e_j) + \gamma) - c(e_i) = w_H \cdot G(h(e_i) - h(e_j) - \gamma) - c(e_i)$$

with $G(h(e_i) - h(e_j) - \gamma)$ as i's likelihood of winning by gap γ ($j = 1, 2$; $j \neq i$). From the first-order conditions we can see that in the symmetric pure-strategy equilibrium each agent's effort choice is identical to that in the basic model with i.i.d. luck for a given tournament contract $(w_H, w_L = 0, \gamma)$:[22]

$$w_H \cdot g(\gamma) = \frac{c'(e)}{h'(e)}. \qquad (16)$$

[21] For example, if the principal is an employer who hires two workers as agents, we can imagine that a union and the employer had signed a general collective agreement that restricts performance standards (e.g., quotas for bonus contracts and gaps for tournament contracts) to a certain upper bound $\bar{\gamma} > 0$.

[22] Recall again that g is symmetric about zero. Hence, (16) is identical to (3).

Thus, the agents' strategic problem does not depend on whether the principal withholds the winner prize if neither agent has won with distance of at least γ.

However, the principal's problem at stage 1 strictly differs from that in the basic model, since now he has to pay w_H only with probability

$$P\left(x_1\left(e_1\right) > x_2\left(e_2\right) + \gamma \text{ or } x_2\left(e_2\right) > x_1\left(e_1\right) + \gamma\right) = 2G\left(-\gamma\right) = 2\left[1 - G\left(\gamma\right)\right]. \tag{17}$$

The principal solves

$$\max_{e,\gamma,w_H} 2h\left(e\right) - w_H P\left(x_1\left(e_1\right) > x_2\left(e_2\right) + \gamma \text{ or } x_2\left(e_2\right) > x_1\left(e_1\right) + \gamma\right) \text{ s.t. } (16), (17)$$
$$= \max_{e,\gamma} 2 \cdot \left(h\left(e\right) - \frac{c'\left(e\right)}{h'\left(e\right)} \frac{1 - G\left(\gamma\right)}{g\left(\gamma\right)}\right). \tag{18}$$

Contrary to the basic model, now the principal faces the following trade-off when choosing the optimal gap: On the one hand, he should choose a very large gap to minimize the probability of paying out the winner prize and, hence, to minimize expected costs for implementing a certain effort level (*implementation-cost effect*). On the other hand, a very large gap may also minimize incentives (*incentive effect*). If, for example, the convolution is a normal distribution, $g\left(\gamma\right)$ will monotonically fall from the mean to each tail.

Let $r := g/\left[1 - G\right]$ denote the hazard rate of the composed random variable $\theta_i - \theta_j$, $\left(e^*, \gamma^*\right)$ the solution to the basic model with i.i.d. luck and $\left(e^{**}, \gamma^{**}\right)$ the solution to problem (18). Then, we obtain the following result:

Theorem 12 *a) If r is monotonically decreasing on $[0, \bar{\gamma}]$, then $\gamma^{**} = 0$ and the principal implements effort $e^{**} = e^*$ characterized by (5). b) If r is monotonically increasing on $[0, \bar{\gamma}]$, then $\gamma^{**} = \bar{\gamma}$ and $e^{**} > e^*$. c) Log-concavity of g implies $\gamma^{**} = \bar{\gamma}$ and $e^{**} > e^*$.*

Problem (18) shows that the principal wants to maximize the value of the hazard rate to trade off the implementation-cost effect against the incentive effect. In principle, the shape of the hazard rate r on the interval $[0, \bar{\gamma}]$ is not clear. It may be decreasing, increasing, bathtub shaped, upside-down bathtub shaped or even constant.[23] However, if it is monotonic we will obtain a clear-cut result: In case of a monotonically decreasing (increasing) hazard rate the principal forgoes a gap (chooses the maximum possible gap). In the former case, the principal implements the same effort level as in the basic model. The objective functions (4) and (18) clearly differ, but since the incentive effect dominates the implementation-cost effect so that $P(x_1(e_1) > x_2(e_2) + \gamma \text{ or } x_2(e_2) > x_1(e_1) + \gamma) = 1$ in Theorem 12a) (i.e., the agents get the winner prize for sure) the two solutions coincide.

As we know from the proof of Theorem 2a), the convolution has a peak at zero, which is also the global maximum of g. If the convolution is single-peaked, the hazard rate cannot be monotonically decreasing over the full domain: Up to the peak, $r = g/[1 - G]$ is monotonically increasing since the numerator is increasing and the denominator decreasing. Right to the peak the hazard rate will be still increasing if the decreasing denominator dominates the decreasing numerator. This strict dominance of the implementation-cost effect over the incentive effect holds for the class of log-concave densities g (e.g., for the normal distribution).[24] For these distributions, the principal prefers $\gamma^{**} = \bar{\gamma}$ to minimize the probability of paying out the winner prize, $2[1 - G(\gamma)]$. Since $g(\gamma)$ – and, hence, agents' incentives – become smaller the larger the gap γ the principal has to compensate for the incentive effect by choosing an appropriately large winner prize w_H according to (16). In contrast to the setting in Section 4, here the principal does not choose a positive gap in order to maximize incentives. On the contrary, he minimizes incentives by the gap in order to minimize expected implementation costs as

[23] See, e.g., Glaser (1980). As example for the last case, suppose that θ_1 and θ_2 are independent and have the same exponential distribution. The corresponding convolution will be a Laplace density (see Feller 1966, pp. 48-49, Ross 2003, p. 333), with a hazard rate that is monotonically increasing over the negative domain and constant over the positive domain. Of course, in that case the principal is indifferent between any feasible gap and not imposing a gap at all.

[24] See Bagnoli and Bergstrom (2005), section 6, on further examples. Recall that if θ is normally distributed, the convolution g will again be a normal distribution.

well.

There are also cases where the incentive effect and the implementation-cost effect together lead to an interior solution $\gamma^{**} \in (0, \bar{\gamma})$ for the optimal gap. Let g be single-peaked but not log-concave. Single-peakedness implies that r is strictly increasing in the negative domain. For positive values the convolution g is strictly decreasing so that r may switch from increasing to decreasing with an interior maximum in the interval $(0, \bar{\gamma})$.[25] If such an interior solution γ^{**} exists it will be described by the first-order condition $r'(\gamma^{**}) = \left[g'(\gamma^{**})[1 - G(\gamma^{**})] + g(\gamma^{**})^2\right] / [1 - G(\gamma^{**})]^2 = 0 \Leftrightarrow -g'(\gamma^{**}) / g(\gamma^{**}) = r(\gamma^{**})$, which can only be satisfied for strictly positive values. In this case, the principal again chooses an optimal gap that is positive (i.e., $\gamma^{**} > 0$).

In the cases of a monotonically increasing hazard rate and a hazard rate with an interior maximum on $(0, \bar{\gamma})$, the principal implements a strictly higher effort compared to the basic model (i.e., $e^{**} > e^{*}$), since e^{**} increases in $r(\gamma^{**})$. In this situation, the principal prefers to decrease incentives by a positive gap in order to reduce expected implementation costs. Missing incentives are then offset by a sufficiently large winner prize. Thus, as $e^{**} > e^{*}$, introduction of a gap is positively related to higher effort but it does *not induce* this high effort level. Finally, if the principal implements more than first-best effort in the basic model, this inefficiency will be even aggravated under verifiable performance signals.

6.2 Comparison with Bonus Contracts

In the given setting with verifiable individual performance signals, the principal is able to apply other incentive schemes than tournaments. Kim (1997) and Oyer (2000) pointed out that, under limited liability, bonus contracts dominate other individual incentive schemes. Bonuses may even lead to the

[25] Examples for single-peaked densities and hazard rates with an interior maximum are given by the log-normal distribution, the inverse Gaussian distribution and the log-logistic distribution; see Glaser (1980), p. 670, Kalbfleisch and Prentice (2002), pp. 34-38.

first-best solution under certain conditions. In this subsection, we compare tournaments with bonuses to check whether, from the principal's point of view, incentives under a bonus contract can be even improved when using competition between agents via a tournament. Analogously to Kim (1997) and Oyer (2000), we assume that $\theta_1, \theta_2 \in [0, \infty)$, which excludes the realization of negative outputs.

The bonus contract $\left(\bar{b}, b, \hat{\gamma}\right)$ consists of three elements: The principal offers a non-zero base wage $\bar{b}$ independent of the agent's performance. Just like the optimal loser prize, the base wage of the optimal bonus contract is zero. In addition, the agent receives a bonus $b \geq 0$ if realized output (1) exceeds a certain threshold or quota $\hat{\gamma}$. In analogy to the tournament, we restrict the principal's choice to values $\hat{\gamma} \in [0, \bar{\gamma}]$ with $\bar{\gamma} > 0$.

Agent i's ($i = 1, 2$) expected utility is given by

$$b \cdot P\left(x_i\left(e_i\right) \geq \hat{\gamma}\right) - c\left(e_i\right) = b \cdot \left[1 - F\left(\hat{\gamma} - h\left(e_i\right)\right)\right] - c\left(e_i\right)$$

with F as cdf and f as density corresponding to the i.i.d. random variables θ_1 and θ_2. As Oyer (2000) we assume that the agent's objective function is well-behaved and that optimal effort choice can be described by the first-order condition

$$b \cdot h'\left(e_i\right) f\left(\hat{\gamma} - h\left(e_i\right)\right) - c'\left(e_i\right) = 0 \Leftrightarrow b = \frac{c'\left(e_i\right)}{h'\left(e_i\right) f\left(\hat{\gamma} - h\left(e_i\right)\right)}. \tag{19}$$

Thus, the principal solves

$$\begin{aligned} &\max_{b \geq 0, \hat{\gamma}, e_i} h\left(e_i\right) - b \cdot P\left(x_i\left(e_i\right) \geq \hat{\gamma}\right) \qquad \text{s.t. } (19) \\ &= \max_{\hat{\gamma}, e_i} h\left(e_i\right) - \frac{c'\left(e_i\right)}{h'\left(e_i\right)} \frac{1 - F\left(\hat{\gamma} - h\left(e_i\right)\right)}{f\left(\hat{\gamma} - h\left(e_i\right)\right)}. \end{aligned} \tag{20}$$

Note that problems (18) and (20) look similar because in each case the principal wants to maximize a hazard rate. However, these hazard rates belong to different distributions. In case of the optimal bonus contract, the principal cares for the hazard rate of θ_i ($i = 1, 2$) based on the distribution with density f. Let $r_\theta := f / \left[1 - F\right]$ denote this hazard rate. In general, a comparison of

r and r_θ does not yield a clear-cut result whether the optimal tournament with a gap dominates the optimal bonus contract or vice versa. However, for a certain family of distributions we obtain the following result:

Theorem 13 *If f is log-concave, the optimal tournament with a gap will dominate the optimal bonus contract.*

The theorem points out that for a wide class of probability distributions, including the normal distribution, it is optimal for a principal to apply a tournament with a gap instead of a bonus contract. Under the convolution g the probability mass of 1 is distributed with doubled variance compared to the initial distribution for θ_i, which makes the density g flatter relative to f and, hence, the peak $\max g$ smaller than the peak $\max f$. Thus, for given monetary payments the bonus contract has an incentive advantage over the tournament contract with a gap. However, this aspect is not essential as payments can be appropriately adjusted and at the optimum the principal will not take the peak of the convolution. Instead, the principal profits from paying out the winner prize w_H with probability strictly less than one half (i.e., $1 - G(\gamma^{**}) < 1/2$ due to the symmetry of g).

7 Conclusion

Our results have shown that in the standard tournament model with i.i.d. luck and limited liability it is never optimal to complement a tournament contract with a positive gap in order to increase agents' incentives. However, a positive gap can be useful (1) to partially insure the agents against income risk, (2) to restore marginal incentives under heterogeneity and (3) to reduce expected implementation costs under verifiable performance signals. In the latter case, the optimal tournament with a gap will dominate the optimal bonus contract if exogenous noise follows a log-concave distribution.

Appendix

Proof of Theorem 2:

a) If θ_1 and θ_2 are i.i.d. with density f the convolution g will be symmetric about zero, so that in view of Theorem 1 the principal maximizes $g(-\gamma) + g(\gamma) = 2g(\gamma)$. Applying the Cauchy-Schwarz inequality leads to[26]

$$\begin{aligned} g(\gamma) &= \int_{-\infty}^{\infty} f(\theta) f(\theta-\gamma)\, d\theta \leq \sqrt{\int_{-\infty}^{\infty} [f(\theta)]^2\, d\theta} \sqrt{\int_{-\infty}^{\infty} [f(\theta-\gamma)]^2\, d\theta} \\ &= \int_{-\infty}^{\infty} [f(\theta)]^2\, d\theta = g(0) \qquad \text{for all } \gamma, \end{aligned}$$

so that $\gamma = 0$ is an optimal gap. To prove uniqueness suppose that $g(\gamma) = g(0)$ for some $\gamma > 0$. Then there must hold equality in the Cauchy-Schwarz inequality, which implies that there is a constant $C > 0$ such that $f(\theta) = Cf(\theta-\gamma)$ for almost all θ. As $\int f(\theta)\, d\theta = 1 = \int f(\theta-\gamma)\, d\theta$, $C = 1$. It follows that for every $a \in \mathbb{R}$, $\int_{a-\gamma}^{a} f = \int_{a}^{a+\gamma} f$, which implies that $\int_{-\infty}^{\infty} f \in \{0, \infty\}$. This is impossible, and it follows that $\gamma = 0$ is the unique optimal gap.

b)–d) Under each of the conditions b) to d), g is also symmetric. We will show that g is unimodal. It follows that g attains its maximum at 0, so that a zero gap is optimal.

According to a result due to Wintner, the convolution of two symmetric unimodal densities is unimodal, see Dharmadhikari and Joag-dev (1988), p. 13. Thus under condition b), g is indeed unimodal. For c) suppose $-\theta_2$ has a log-concave density and θ_1 has a unimodal density. Then, by a result due to Ibragimov, $-\theta_2$ is strongly unimodal and, therefore, g is unimodal, see Dharmadhikari and Joag-dev (1988), pp. 17-23. If θ_1 and θ_2 have a log-concave joint density, then any non-degenerate linear combination of θ_1 and θ_2, in particular $\theta_1 - \theta_2$, has a log-concave density, see Dharmadhikari and Joag-dev (1988) pp. 47-51. This shows that g is unimodal under d) as well. □

[26] See, e.g., Mood et al. (1974), pp. 185-186, for the convolution formula. The step from line 1 to line 2 uses the fact that $\int_{\underline{x}}^{\bar{x}} y(x-\alpha)\, dx = \int_{\underline{x}-\alpha}^{\bar{x}-\alpha} y(x)\, dx$.

Proof of Proposition 3:

Recall that $\gamma^* \in \arg\max_{\gamma}\{g(-\gamma)+g(\gamma)\}$ and $w_L^* = 0$. Thus, by inserting the incentive constraint

$$w_H = \frac{2c'(e)}{[g(-\gamma^*)+g(\gamma^*)]h'(e)}$$

into the principal's objective function (4), his optimization problem can be summarized as

$$\max_e 2\left(h\left(e\right)+E\left[\theta_1+\theta_2\right]\right) - \frac{2c'(e)}{[g(-\gamma^*)+g(\gamma^*)]h'(e)}.$$

Since $h\left(e\right)$ is concave and $c'\left(e\right)/h'\left(e\right)$ strictly convex (see (2)), the principal implements the effort level that is described by the first-order condition, which can be rearranged to

$$h'\left(e\right)\left[g(-\gamma^*)+g(\gamma^*)\right] = \frac{\partial}{\partial e}\left(\frac{c'\left(e\right)}{h'\left(e\right)}\right) \Leftrightarrow g(-\gamma^*)+g(\gamma^*) = \frac{c''\left(e\right)h'\left(e\right)-c'\left(e\right)h''\left(e\right)}{\left[h'\left(e\right)\right]^3}.$$

□

Proof of Theorem 4:

Let μ denote the distribution of $\eta_1 - \eta_2$. Let ψ denote the density of $\epsilon_1 - \epsilon_2$; that is, $\psi(y) = \int f(x)f(x+y)\,dx$. Then the density of $\eta_1 + \sigma\epsilon_1 - \eta_2 - \sigma\epsilon_2$ is given by

$$g_\sigma(y) = \frac{1}{\sigma}\int \psi\left(\frac{y-x}{\sigma}\right)\,d\mu(x),$$

see e.g. Billingsley (1995), p. 266. Let

$$H_\sigma(y) := \sigma[g_\sigma(\sigma y) + g_\sigma(-\sigma y)].$$

In view of Theorem 1, we have to show that $H_\sigma(0) > H_\sigma(y)$ for all $y \neq 0$ when σ is large enough.

Since f satisfies a Lipschitz condition, f is absolutely continuous. It follows that f is differentiable almost everywhere and that f' is essentially bounded. Using that f is integrable and satisfies a Lipschitz condition, one may verify by an application of Lebesgue's dominated convergence theorem that for every $y \in \mathbb{R}$,

$$\psi'(y) = \int f(x)f'(x+y)\,dy = \int f(x-y)f'(x)\,dx.$$

In particular, ψ' is bounded. Using now that f' is integrable, one may show similarly that for every $y \in \mathbb{R}$,

$$\psi''(y) = -\int f'(x-y)f'(x)\,dx.$$

It follows that ψ'' is bounded and continuous, see Hewitt and Stromberg (1969), p. 398. We have $\psi''(0) = -\int [f'(x)]^2 dx$. If $\int [f'(x)]^2 dx = 0$, then $f' = 0$ almost everywhere, and as f is absolutely continuous, it would follow that f is constant. This is impossible since f is a probability density. Hence $\psi''(0) < 0$. Thus there exists $x_0 > 0$ such that $\psi''(x) \leq \psi''(0)/2$ for all $x \in [-x_0, x_0]$. If $|y| \leq x_0/2$, then

$$\begin{aligned} H''_\sigma(y) &= \left\{ \int_{[-x_0\sigma/2, x_0\sigma/2]} + \int_{\mathbb{R}\setminus[-x_0\sigma/2, x_0\sigma/2]} \right\} \psi''\left(y - \frac{x}{\sigma}\right) + \psi''\left(-y - \frac{x}{\sigma}\right)\,d\mu(x) \\ &\leq \mu\left(\left[-\frac{x_0\sigma}{2}, \frac{x_0\sigma}{2}\right]\right)\psi''(0) + 2\left\{1 - \mu\left(\left[-\frac{x_0\sigma}{2}, \frac{x_0\sigma}{2}\right]\right)\right\} \sup_{x\in\mathbb{R}} \psi''(x), \end{aligned}$$

where we used that ψ' and ψ'' are bounded. It follows that for σ sufficiently large, H_σ is strictly concave on $[-x_0/2, x_0/2]$, so that $H_\sigma(0) > H_\sigma(y)$ for all $y \in [-x_0/2, x_0/2] \setminus \{0\}$.

Let

$$M := \sup\{\psi(y-x) + \psi(-y-x) : x \in \mathbb{R}, y \in \mathbb{R} \setminus (-x_0/2, x_0/2)\}.$$

An application of the Cauchy-Schwarz inequality shows that $\psi(y) < \psi(0)$ for all $y \neq 0$. Since f is square-integrable, ψ is continuous and $\lim_{x\to-\infty}\psi(y) = \lim_{y\to\infty}\psi(y) \to 0$, see Hewitt and Stromberg (1969), p. 398. It follows that $M < 2\psi(0)$. For all y with $|y| > x_0/2$ and all $\sigma > 0$,

$$H_\sigma(y) = \int \psi\left(y - \frac{x}{\sigma}\right) + \psi\left(-y - \frac{x}{\sigma}\right)\,d\mu(x) \leq M.$$

On the other hand, by dominated convergence, $H_\sigma(0) \to 2\psi(0)$ as $\sigma \to \infty$, so that $H_\sigma(0) > M$ for σ sufficiently large. □

Proof of Proposition 5:

The density of $\eta_1 + \sigma\epsilon_1 - \eta_2 - \sigma\epsilon_2$ is given by $g(y) = \sigma^{-1}\int \psi((y-x)/\sigma)\, d\mu(x)$, where μ is the distribution of $\eta_1 - \eta_2$ and ψ is the density of $\epsilon_1 - \epsilon_2$.

a) The convexity assumption on ψ implies that for every $x \neq 0$, $\psi(x+y) + \psi(x-y)$ is strictly increasing in y for $y \in [0, |x|]$. Hence if $0 \leq y_1 < y_2 \leq C$, then, by (8),

$$\begin{aligned} g(y_1) + g(-y_1) &= \frac{1}{\sigma}\int \psi\left(\frac{y_1 - x}{\sigma}\right) + \psi\left(\frac{-y_1 - x}{\sigma}\right) d\mu(x) \\ &< \frac{1}{\sigma}\int \psi\left(\frac{y_2 - x}{\sigma}\right) + \psi\left(\frac{-y_2 - x}{\sigma}\right) d\mu(x) = g(y_2) + g(-y_2). \end{aligned}$$

That is, $g(y) + g(-y)$ is strictly increasing on $[0, C]$, so that every optimal gap must belong to $[C, \infty)$.

b) Here $\epsilon_1 - \epsilon_2$ has a Laplace distribution with density $\psi(x) = e^{-|x|}/2$. If $|x| \leq |y|$, then

$$\frac{\psi(x+y) + \psi(x-y)}{2\psi(x)} = \frac{e^{-|y|} + e^{2|x|-|y|}}{2} \geq e^{-|y|} \geq 1 - |y|,$$

and if $|x| \geq |y|$, then

$$\frac{\psi(x+y) + \psi(x-y)}{2\psi(x)} = \cosh y = \sum_{k=0}^{\infty} \frac{y^{2k}}{(2k)!} \geq 1 + \frac{y^2}{2}.$$

Hence for all $y > 0$,

$$\begin{aligned} g(y) + g(-y) &= \frac{1}{\sigma}\left\{\int_{(-y,y)} + \int_{(-y,y)^C}\right\} \psi\left(\frac{y-x}{\sigma}\right) + \psi\left(\frac{-y-x}{\sigma}\right) d\mu(x) \\ &\geq \frac{2}{\sigma}\left(1 - \frac{y}{\sigma}\right)\int_{(-y,y)} \psi\left(\frac{x}{\sigma}\right) d\mu(x) + \frac{2}{\sigma}\left(1 + \frac{y^2}{2\sigma^2}\right)\int_{(-y,y)^C} \psi\left(\frac{x}{\sigma}\right) d\mu(x) \\ &= 2g(0) - \frac{2y}{\sigma^2}\int_{(-y,y)} \psi\left(\frac{x}{\sigma}\right) d\mu(x) + \frac{y^2}{\sigma^3}\int_{(-y,y)^C} \psi\left(\frac{x}{\sigma}\right) d\mu(x) \\ &\geq 2g(0) + \frac{y}{\sigma^2}\left[-\mu((-y,y)) + \frac{y}{\sigma}\int_{(-y,y)^C} \psi\left(\frac{x}{\sigma}\right) d\mu(x)\right]. \end{aligned}$$

By assumption (9), the term in brackets is positive for all sufficiently small positive y. Therefore, every optimal gap must be positive. □

Proof of Proposition 7:
The proof of Proposition 7 rests on the following inequality for the normal density $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$.

Lemma 14 *If* $a \in [-1, 1]$, *then*

$$\phi(x+a) + \phi(x-a) < 2\phi(a) \qquad \text{for all } x \neq 0.$$

Proof. Fix $a \in [-1, 1]$ and set $\psi(x) = \phi(x+a) + \phi(x-a)$. Then

$$\psi'(x) = \frac{1}{\sqrt{2\pi}} e^{-(x^2+a^2)/2} \psi_1(x),$$

where $\psi_1(x) = (a-x)e^{ax} - (a+x)e^{-ax}$. Since

$$\psi_1'(x) = (a^2-1)\left(e^{ax} + e^{-ax}\right) - ax\left(e^{ax} - e^{-ax}\right) < 0$$

for all $x \neq 0$ and $\psi_1(0) = 0$, it follows that $\psi'(x) > 0$ for $x < 0$ and $\psi'(x) < 0$ for $x > 0$. Hence $\psi(x) < \psi(0) = 2\phi(a)$ for all $x \neq 0$. ■

Proposition 7 can now be proved as follows. Let μ denote the distribution of $\eta_1 - \eta_2$. The density of $\sigma(\epsilon_1 - \epsilon_2)$ is $\phi(x/(\sqrt{2}\sigma))/(\sqrt{2}\sigma)$, and so the density of $\eta_1 + \sigma\epsilon_1 - \eta_2 - \sigma\epsilon_2$ is given by

$$g(x) = \frac{1}{\sqrt{2}\sigma} \int \phi\left(\frac{x-y}{\sqrt{2}\sigma}\right) d\mu(y).$$

Let $\overline{g}(x) = g(x) + g(-x)$ for all $x \geq 0$. If $|y| \leq \sqrt{2}\sigma$, then, by Lemma 14,

$$\phi\left(\frac{x-y}{\sqrt{2}\sigma}\right) + \phi\left(\frac{-x-y}{\sqrt{2}\sigma}\right) < 2\phi\left(\frac{y}{\sqrt{2}\sigma}\right)$$

for all $x \in \mathbb{R} \setminus \{0\}$. Hence, if $|\eta_1 - \eta_2| \leq \sqrt{2}\sigma$ almost surely, then for all $x > 0$,

$$\overline{g}(x) = \frac{1}{\sqrt{2}\sigma} \int_{[-\sqrt{2}\sigma, \sqrt{2}\sigma]} \phi\left(\frac{x-y}{\sqrt{2}\sigma}\right) + \phi\left(\frac{-x-y}{\sqrt{2}\sigma}\right) d\mu(y) < \overline{g}(0),$$

so that, by Theorem 1, the optimal gap is zero.

On the other hand, if $|\eta_1 - \eta_2| > \sqrt{2}\sigma$ almost surely, then $\overline{g}'(0) = 0$ and

$$\begin{aligned}\overline{g}''(0) = 2g''(0) &= \frac{1}{\sqrt{2}\sigma^3} \int \phi'' \left(\frac{-y}{\sqrt{2}\sigma} \right) d\mu(y) \\ &= \frac{1}{\sqrt{2}\sigma^3} \int \phi \left(\frac{y}{\sqrt{2}\sigma} \right) \left(\frac{y^2}{2\sigma^2} - 1 \right) d\mu(y) > 0.\end{aligned}$$

Differentiation under the integral sign is justified because ϕ' and ϕ'' are bounded. It follows that every optimal gap is positive. [By bounded convergence, $\lim_{x\to\infty} \overline{g}(x) = 0$, and it follows that $\overline{g}$ attains its maximum in $(0, \infty)$.]
□

Proof of Proposition 9:
The proof uses the following auxiliary result for the normal density evaluated at three equidistant points.

Lemma 15 *Let $a \in \mathbb{R}$ and*

$$\psi(x) = \phi(x - a) + \phi(x) + \phi(x + a).$$

Then $\psi(x) < \psi(0)$ for all $x \neq 0$.

Proof. As ϕ is strictly decreasing on $[0, \infty)$, ψ is strictly decreasing on $[a, \infty)$, so that $\psi(x) < \psi(a)$ for all $x > a$. If $x \in (a/2, a]$, then $a - x \in [0, a/2)$ and $0 \leq 2a - x < x + a$, so that

$$\psi(a - x) = \psi(x) + \phi(2a - x) - \phi(x + a) > \psi(x).$$

Hence $\psi(x) < \max_{y\in[0,a/2]} \psi(y)$ for all $x > a/2$. We will now show that ψ is strictly decreasing on $[0, a/2]$. It will then follow that $\psi(0) > \psi(x)$ for all $x > 0$ and so, as $\psi(x) = \psi(-x)$, $\psi(0) > \psi(x)$ for all $x \neq 0$. We have

$$\psi'(x) = \phi(x)\, e^{-a^2/2} s(x),$$

where

$$s(x) = -xe^{a^2/2} + 2a \sinh ax - 2x \cosh ax = \sum_{k=0}^{\infty} c_k x^{2k+1}$$

and

$$c_0 = -e^{a^2/2} + 2(a^2-1), \qquad c_k = 2\frac{a^{2k}}{(2k)!}\left(\frac{a^2}{2k+1} - 1\right), \qquad k \geq 1.$$

For every a,

$$\begin{aligned} c_0 &< -\sum_{k=0}^{3} \frac{1}{k!}\left(\frac{a^2}{2}\right)^k + 2(a^2-1) \\ &= \frac{7\sqrt{7}-19}{3} - \frac{(a^2+2+4\sqrt{7})(a^2+2-2\sqrt{7})^2}{48} < 0. \end{aligned}$$

If $a^2 < 3$, then $c_k < 0$ for all $k \geq 0$, so that $\psi'(x) < 0$ for all $x > 0$.
Suppose next that $3 \leq a^2 \leq 4$. Then $c_1 \geq 0$ and $c_k \leq 0$ for all $k \geq 2$. Hence for all $x \in (0, a/2]$,

$$\frac{s(x)}{x} \leq c_0 + c_1\frac{a^2}{4} \leq -\sum_{k=0}^{6} \frac{a^{2k}}{2^k k!} - 2 + a^2 + \frac{a^4}{3} =: w(a),$$

say. An application of Descartes' rule of sign [Pólya and Szegö (1976), p. 41] shows that w' has exactly one positive zero, and as $w'(2) = 2/15 > 0$ and $\lim_{x\to\infty} w'(x) = -\infty$, w' must be strictly positive on $(0, 2)$. Hence for all $x \in (0, a/2]$, $s(x)/x \leq w(a) \leq w(2) = -1/45 < 0$.
Suppose finally that $a^2 > 4$. Then $c_k > 0$ if $1 \leq k < (a^2-1)/2$ and $c_k < 0$ if $k > (a^2-1)/2$. By the version of Descartes' rule of sign for series, s' has at most two positive zeros. As $s'(0) = c_0 < 0$, and

$$s'\left(\frac{a}{2}\right) = \left(-2 + \frac{a^2}{2}\right)e^{a^2/2} + \left(\frac{3a^2}{2} - 1\right)e^{-a^2/2} > 0,$$

s' has exactly one zero in $(0, a/2]$ and s' changes sign from minus to plus at this zero. Hence for all $x \in (0, a/2)$,

$$s(x) < \max\left\{s(0), s\left(\frac{a}{2}\right)\right\} = \max\left\{0, -\frac{3}{2}ae^{-a^2/2}\right\} = 0.$$

■

The proof of Proposition 9 proceeds as follows. To determine the distribution of $\eta_1 - \eta_2$ note first that

$$\varrho = \frac{E(\eta_1\eta_2) - [E(\eta_1)]^2}{Var(\eta_1)} = \frac{P(\eta_1 = \eta_2 = a) - p^2}{p - p^2},$$

so that $P(\eta_1 = \eta_2 = a) = p^2 + (p - p^2)\varrho$. Hence

$$P(\eta_1 = a, \eta_2 = 0) = P(\eta_1 = 0, \eta_2 = a) = P(\eta_1 = a) - P(\eta_1 = \eta_2 = a)$$
$$= (p - p^2)(1 - \varrho).$$

It follows that the density of $\eta_1 + \sigma\epsilon_1 - \eta_2 - \sigma\epsilon_2$ is given by

$$\begin{aligned} g(x) &= \sum_{y \in \{-a,0,a\}} \frac{1}{\sqrt{2}\sigma}\phi\left(\frac{x-y}{\sqrt{2}\sigma}\right) P(\eta_1 - \eta_2 = y) \\ &= \frac{(p-p^2)(1-\varrho)}{\sqrt{2}\sigma}\psi(x) + \frac{1 - 3(p-p^2)(1-\varrho)}{\sqrt{2}\sigma}\phi\left(\frac{x}{\sqrt{2}\sigma}\right), \end{aligned}$$

where

$$\psi(x) = \phi\left(\frac{x-a}{\sqrt{2}\sigma}\right) + \phi\left(\frac{x}{\sqrt{2}\sigma}\right) + \phi\left(\frac{x+a}{\sqrt{2}\sigma}\right).$$

By Lemma 15, $\psi(x) < \psi(0)$ for all $x \neq 0$. Thus, if $(p - p^2)(1 - \varrho) \leq \frac{1}{3}$, then $g(x) < g(0)$ for all $x \neq 0$, and so the optimal gap must be zero. If $a \leq \sqrt{2}\sigma$, then $P(|\eta_1 - \eta_2| \leq \sqrt{2}\sigma) = 1$, and the optimal gap is zero by Proposition 7. Suppose next that $(p - p^2)(1 - \varrho) > \frac{1}{3}$ and that a satisfies inequality (11). Then

$$\begin{aligned} g(a) - g(0) &> \frac{(p-p^2)(1-\varrho)}{\sqrt{2}\sigma}\phi(0) - g(0) \\ &= \frac{1}{2\sigma\sqrt{\pi}}\left[(p-p^2)(1-\varrho)\left(3 - 2e^{-a^2/(4\sigma^2)}\right) - 1\right] \\ &\geq 0. \end{aligned}$$

Hence $g(a) + g(-a) = 2g(a) > 2g(0)$, so that a zero gap is not optimal. Since $\lim_{x\to\infty} g(x) + g(-x) = 0$, there exists an optimal gap. To prove uniqueness note that $g'(x)$ can be written in the form

$$g'(x) = \phi\left(\frac{x}{\sqrt{2}\sigma}\right)[p_1(x)e^{-ax/(2\sigma^2)} + p_2(x) + p_3(x)e^{ax/(2\sigma^2)}],$$

where p_1, p_2, p_3 are affine functions. The term in brackets is an exponential polynomial of degree 5, so that g' has at most 5 real zeros, see e.g. Hirschman and Widder (1955), p. 31. Therefore, as g is even, g can have at most two local extrema in $(0, \infty)$ and only one of them can be a local maximum. This shows that the optimal gap is unique. □

Proof of Theorem 12:

In order to minimize expected implementation costs for a certain effort level e, the principal will always choose the gap γ that maximizes $r(\gamma)$. Then for given γ^{**}, the principal implements optimal effort e^{**} that solves (18) by fine-tuning incentives via w_H.

a) If $r'(\gamma) < 0$, $\forall \gamma \in [0, \bar{\gamma}]$, then $\gamma^{**} = 0$ is optimal, leading to $r(0) = g(0) / [1 - G(0)] = 2g(0)$. Inserting into (18) and computing the first-order condition for the optimal effort leads to the same result as in the basic model, described by (5).

b) If $r'(\gamma) > 0$, $\forall \gamma \in [0, \bar{\gamma}]$, then $\gamma^{**} = \bar{\gamma}$. The first-order condition to problem (18) shows that optimal effort is implicitly described by

$$r(\gamma) = \frac{c''(e) h'(e) - c'(e) h''(e)}{[h'(e)]^3}.$$

Concavity of $h(e)$ and condition (2) imply that effort e is monotonically increasing in $r(\gamma)$. Since $\gamma^{**} = \bar{\gamma}$ and $r(\bar{\gamma}) > r(0) = 2g(0)$ the principal implements effort $e^{**} > e^*$.

c) An (1998) and Bagnoli and Bergstrom (2005) show that log-concavity of a density function implies that this density is unimodal and has a monotonically increasing hazard rate. Applying the finding of result b) immediately leads to $\gamma^{**} = \bar{\gamma}$ and $e^{**} > e^*$. □

Proof of Theorem 13:

We can show that any effort level $e > 0$ can be implemented at less cost under a tournament with a gap compared to a bonus contract. According to (18) and (20), this claim is true if[27]

$$\frac{g(\gamma^{**})}{1 - G(\gamma^{**})} > \frac{f(\hat{\gamma} - h(e))}{1 - F(\hat{\gamma} - h(e))} \Leftrightarrow r(\gamma^{**}) > r_\theta(\hat{\gamma} - h(e))$$

[27] Note that (18) describes the principal's maximization problem for both agents.

for all $e > 0$ and all $\hat{\gamma} \in [0, \bar{\gamma}]$. Since f is log-concave, r_θ is monotonically increasing (An (1998) and Bagnoli and Bergstrom (2005)). In that case, using the convolution formula $g(\alpha) = \int_0^\infty f(\alpha + \theta) f(\theta) d\theta$ (e.g., Mood et al. 1974, p. 185) we can show that $r(\alpha) > r_\theta(\alpha)$, $\forall \alpha$:[28]

$$\begin{aligned}
r(\alpha) = \frac{g(\alpha)}{1 - G(\alpha)} &= \frac{\int_0^\infty f(\alpha+\theta) f(\theta) d\theta}{1 - \int_0^\infty F(\alpha+\theta) f(\theta) d\theta} \\
&= \frac{\int_0^\infty r_\theta(\alpha+\theta)\left[1 - F(\alpha+\theta)\right] f(\theta) d\theta}{\int_0^\infty \left[1 - F(\alpha+\theta)\right] f(\theta) d\theta} \\
&> \frac{\int_0^\infty r_\theta(\alpha)\left[1 - F(\alpha+\theta)\right] f(\theta) d\theta}{\int_0^\infty \left[1 - F(\alpha+\theta)\right] f(\theta) d\theta} \quad \text{since } r'_\theta(\theta) > 0, \forall \theta \\
&= r_\theta(\alpha).
\end{aligned}$$

As $\gamma^{**} = \bar{\gamma} > \hat{\gamma} - h(e)$, $\forall e > 0, \hat{\gamma}$, the claim of Theorem 13 must be true. □

[28] See Miravete (2005), p. 1358, on a similar proof for the sum of two random variables, $\xi_i + \xi_j$.

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